An analytic solution of the Lotka Volterra equations
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It is usual to solve nonlinear differential equations with aid of an analog computer, a procedure yielding results that prove to be satisfactory for most purposes. Although, in general, analytic solutions to nonlinear differential equations are hard to obtain, such solutions would, nevertheless, have the advantage that they could be evaluated without a computer.

In connexion with a problem of laser physics we have solved the following system of two coupled nonlinear differential equations of first order 1

\[
\begin{align*}
\dot{x} &= -ax + cy; \quad a > 0, \quad c > 0, \\
y &= -by - dx; \quad b > 0, \quad d > 0,
\end{align*}
\]

(1)
deriving the exact solution as well as a rather simple approximate solution by means of Gröbner's method of Lie series 2. Since we have nowhere made use of the assumption that all coefficients are positive, these solutions are valid for arbitrary real coefficients as well.

To the problems, which may thus be solved, belong also the Lotka-Volterra equations 3

\[
\begin{align*}
\dot{x} &= \epsilon_1 x - xy; \quad \epsilon_1 > 0, \\
y &= -\epsilon_2 y + xy; \quad \epsilon_2 > 0.
\end{align*}
\]

(2)

Although for system (2) the corresponding differential equation for the trajectories in the phase plane can be solved in closed form, explicit solutions \(x(t)\) and \(y(t)\) have to the authors' knowledge nowhere been given. In view of the importance of the Lotka-Volterra equations as one of the basic models in mathematical ecology 4 and for the study of oscillating reactions 5, connected especially with recent discoveries in biochemistry, explicit solutions to (2) should be of particular interest for biochemists and biologists.

Since the details may be found in our cited paper 4, we will give here only the result. The solution to (2) reads:

\[
x(t) = x(0) e^{at} \left[ 1 + \sum_{n=1}^{\infty} \sum_{h=0}^{n-1} \frac{(-a)^h y(0) e^{dh} n^{-h} K_{n,h}(t)}{h!} \right] \tag{3a}
\]

\[
y(t) = y(0) e^{ct} \left[ 1 + \sum_{n=1}^{\infty} \sum_{h=0}^{n-1} \frac{(-c)^h n^{-h} \overline{K}_{n,h}(t)}{h!} \right] \tag{3b}
\]

where

\[
K_{n,h}(t) = \int_0^t e^{-h s} \left( \int_0^s e^{h \tau} \left( \int_0^\tau e^{h \tau'} \left( \int_0^{\tau'} e^{h \tau''} \cdots \left( \int_0^{\tau''} e^{h \tau_1} \right) \cdots \left( \int_0^{\tau_n} e^{h \tau_{n-1}} \right) \right) \cdots \left( \int_0^{\tau_2} e^{h \tau_1} \right) \cdots \left( \int_0^{\tau_1} e^{h \tau_0} \right) \right) \cdots \right) \cdots \right) \cdots \right)
\]

(4)

and

\[
\overline{K}_{n,h}(t) = \frac{1}{n!} \int_0^t e^{-h s} \left( \int_0^s e^{h \tau} \left( \int_0^\tau e^{h \tau'} \left( \int_0^{\tau'} e^{h \tau''} \cdots \left( \int_0^{\tau_n} e^{h \tau_{n-1}} \right) \cdots \left( \int_0^{\tau_2} e^{h \tau_1} \right) \cdots \left( \int_0^{\tau_1} e^{h \tau_0} \right) \right) \cdots \right) \cdots \right) \cdots \right)
\]

(5)

The solutions (3) are rather cumbersome to deal with because of the complicated expressions \(K_{n,h}(t)\) and \(\overline{K}_{n,h}(t)\), although these coefficient functions do not depend on the initial conditions and the integrations involved can be carried out. In practice, however, it will often be sufficient to replace the \((n-1)\) different integrals in (4) by an average integral, \(K_{n,h}\) and \(\overline{K}_{n,h}\), are then approximated to:

\[
K_{n,h}(t) \approx \frac{1}{n!} \left( \frac{1}{\epsilon_1} \right)^{n-1} e^{-h \epsilon_1} S_{n,h} \tag{6a}
\]

and

\[
\overline{K}_{n,h}(t) \approx \frac{1}{n!} \left( \frac{1}{\epsilon_2} \right)^{n-1} e^{-h \epsilon_2} S_{n,h} \tag{6b}
\]

with

\[
S_{n,h} = \sum_{k=1}^{h+1} \left( -1 \right)^{k+h+1} \left( \frac{n+1}{h+1-k} \right) k^n.
\]

(7)

\[3\] V. Volterra, Théorie Mathématique de la lutte pour la vie, Gautiers-Villars, Paris 1931.