Rays and Phases: A Paradox?*

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Dedication

It is an honor to be able to speak in this meeting celebrating George Sudarshan’s birthday; to know him and to discuss physics with him has been a great privilege for me.

The states of a quantum mechanical system are represented by rays in Hilbert space, but interference phenomena, Berry phase, etc., make reference to vectors. We show how to solve this apparent paradox by appropriate use of the vector bundle structure of quantum theory.

1. The states of a quantum mechanical system are represented by rays, i.e., one-dimensional subspaces of a Hilbert space; but the eigenvalue problems, the evolution, collision theory, interference phenomena etc. are all set in a vector-like formulation.

Are both pictures consistent? In particular are phase differences, typical of vector subtraction, and measurable in interference fringes, describable in terms of rays only? This is the question we address in this paper.

Let $H$ be the Hilbert space of vectors, and $\bar{H} = CP(\mathbb{C}) = PH$ the projective space; the map $H \rightarrow \bar{H}$ takes the ray of a vector; it is enough to take normalized vectors mod a phase; in mathematical terms we have the bundles

$$
\eta: \mathbb{C}^* \rightarrow H \rightarrow \{0\} \rightarrow \bar{H}
$$

$$
\iota: \mathbb{C} \rightarrow E \rightarrow \bar{H}
$$

$$
\eta_0: U(1) \rightarrow S(\infty) \rightarrow \bar{H},
$$

where $\eta$ is the definition of projective space, $\eta_0$ the restriction to compact fibre, $S(\infty)$ the (infinite dimensional) sphere of norm-one vectors, and $\iota$ the associated line bundle. Finally the total space $E$ is similar to $H$:

$$
E - \{0\} \approx H - \{0\}.
$$

We shall see that the bundle formalism allows to use rays and vectors, as long as everything will be “fibre-preserving”.

2. We describe now some consequences of the bundle framework. First, physics is described as a geometry, not as algebra; physical states are “points”, related to each other by transformations (symmetry); the projective space $CP(\mathbb{C})$ is just such a geometrical structure; there is no “state zero”, nor can the states be added, etc., as they were elements of a vector space, which is an algebraic structure.

Secondly, there is no superposition principle, in the ordinary sense that state (1) plus state (2) make state (3): you cannot add rays, as you cannot add points. We can think of two “correct” statements of what is usually called the “superposition principle”:

1) States (1) and (2) define a (complex bi-)plane; any state (ray) in this plane can be considered as a “superposition” of states (1) and (2); with hindsight, this can be seen in Dirac’s book on quantum mechanics.

2) Given a state, it has a well-defined projection in any other state, i.e., the squared modulus of the scalar product of any two normalized vector representatives; in particular, any state has projections in a complete orthonormal set of rays. In this sense one speaks more properly of principle of decomposition of states; this is how Jauch states the (superposition) principle.

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Of course the projections of a state $\psi$ in a base $e_i$, namely $r_i = |\langle \psi | e_i \rangle|^2$ do not fix the state completely; for that one needs to know the relative phases. In fact the overlap between two rays $\varphi$, $\psi$,

$$R(\varphi, \psi) = \frac{\langle \varphi | \psi \rangle}{\| \varphi \| \| \psi \|},$$  \hspace{1cm} (3)

endowes the state space with a metric; in the projective space $CP^1 = S^2$ associated to the complex biplane spanned by $|\varphi\rangle$ and $|\psi\rangle$ we have the metric

$$\text{distance}(\varphi, \psi) = 1 - \text{Tr} g_\varphi g_\psi = \sin^2(\Phi/2), \hspace{1cm} (4)$$

where $g$ are the projectors and $\Phi$ the angle they make in $S^2$, e.g. orthogonal states differ by $1$; however, $\| |\varphi\rangle - |\psi\rangle \| = \sqrt{2}$, if $\langle \varphi | \varphi \rangle = 0$; this $2 \rightarrow 1$ change is due to the Hopf bundle

$$\beta: S^1 \rightarrow S^3 \rightarrow S^2$$  \hspace{1cm} (5)

which should be understood as the “square root” or the “spinor” bundle of the tangent bundle to $S^2$.

As a third consequence of the bundle structure, we remark that wave functions are not, in general, to be identified with complex-valued functions; they are rather sections in the associated line bundle of (1). In particular they can be “multi-valued”; the distinction is due to the non-trivial nature of the bundles (1) and (5), and it is essential to understand the “multi-valuedness” which appears in the Aharonov-Bohm effect, the Berry phase, etc.

As a fourth consequence, in spite of the projective nature of state space one can work with vectors as long as the operations are bundle morphisms, i.e. preserve the fibres; that is, only linear (or antilinear) operators are allowed: this is the true nexus between the geometric character of the states and the algebraic nature of the operations in quantum mechanics.

This means that e.g. eigenvalues are measurable, because the spectral problem is linear,

$$R |\psi\rangle = \lambda |\psi\rangle \Rightarrow A(C |\psi\rangle) = \lambda C |\psi\rangle.$$  \hspace{1cm} (6)

If $A = A^+$, then $\lambda \in \mathbb{R}$, e.g. the energy eigenvalues are observable; if $A^+ = A^{-1}$, $|\lambda| = 1$, e.g. we measure phases as the Berry phase or the dynamical phase, when $A$ is the evolution operator for a closed loop in state space. In this way a unified description of phases as eigenvalues of unitary operators is obtained; if one wishes a purely “projective” characterization of instantaneous measurements, one can write in the above formula

$$\lambda = \text{Tr} A q_\psi \text{ when } [A, q_\psi] = 0 \text{ and } q_\psi = |\psi\rangle \langle \psi|.$$  \hspace{1cm} (7)

3. As an example of motion of the states vs. motion of the vectors, let us look at the simplest quantum-mechanical system, namely a particle in quantum two space dimensions. We have a two-dimensional real vector space $\mathbb{R}^2$; a general state is a projector $g$ and the “hamiltonian” will be antisymmetric,

$$g = \frac{1}{2} (1 + \sigma_3 x + \sigma_1 y) ; \quad H = - i \sigma_2 \hbar;$$

$$x, y, \hbar \in \mathbb{R} ; \quad x^2 + y^2 = 1.$$  \hspace{1cm} (8)

Taking $(x, y) = (\cos \varphi, \sin \varphi)$, $g$ can be written as $|\psi\rangle \langle \psi|$, where

$$g = \begin{pmatrix} \cos^2 \varphi & \cos \varphi \sin \varphi \\ \cos \varphi \sin \varphi & \sin^2 \varphi \end{pmatrix}.$$

$$|\psi\rangle = \pm \begin{pmatrix} \cos \varphi/2 \\ \sin \varphi/2 \end{pmatrix}.$$  \hspace{1cm} (9)

Observe the ambiguity $(\pm)$ and the “halfness” $(\varphi \rightarrow \varphi/2)$. Now the evolution equations

$$\dot{\varphi} = [H, g], \quad g(0) = g(\varphi), \quad |\psi\rangle = H |\psi\rangle$$  \hspace{1cm} (10)

will have the solution $\varphi(t) = \varphi + th/2$, and therefore the states perform a cyclic motion of period $T_0 = 4 \pi/h$ but the vectors will have $T_0 = 8 \pi/2 = 2 T_0$. In particular $|\psi(T_0)\rangle = - |\psi(T_1)\rangle$: this is the simplest case of Berry phase, namely $\pi = 180^\circ$. The spinor or “square root” character of the real bundle

$$\varphi: O(1) = Z_2 \rightarrow S^1 \rightarrow S^1 = RP^1$$  \hspace{1cm} (11)

is the equivalent to (5) in this simpler case; it is well known how the “magnetic problem” associated to (5) will give an angle as Berry phase.

4. To finalize, let us talk about interferences; it is here where the contrast between rays and phases is sharper.

In the typical two-slit experiment, let $\psi_s$ be the state at the source, which is propagated to the screen $P$ by two ways (1) and (2). Let $\langle \psi_\alpha \rangle$ be a representative vector, and let $|\psi_\alpha\rangle$ and $|\psi_\beta\rangle$ be the propagation along paths (1) and (2). The point is now the following: The phase difference of $|\psi_\alpha\rangle$ and $|\psi_\beta\rangle$ is observable because it is a projective concept, i.e. if $|\psi_s\rangle$ were another representative vector of state $\psi_s$, the phase difference between $|\psi_\alpha\rangle$ and $|\psi_\beta\rangle$ is the same as before, because the evolution $S \rightarrow P$ is linear, i.e. projective.
Alternatively, it can be argued that the path $P \rightarrow S \rightarrow P$ is a closed loop, with a total phase shift which is the same interference angle. In fact both arguments are identical; in their second form it has been also put forward by Aharonov and Anandan.

In conclusion, there is no paradox in the use of states as rays and vector representatives, as long as everything is fibre preserving = projective invariant.