Uncertainty Relation: From Inequality to Equality*

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The uncertainty area \( \delta(p, q) := \sqrt{\langle W(p, q)^2 \rangle} \) is proposed in place of \( \delta p \cdot \delta q \) and it is shown that each pure quantum state is a minimum uncertainty state in this sense: \( \delta(p, q) = 2 \pi \hbar \). For mixed states, on the other hand, \( \delta(p, q) > 2 \pi \hbar \). In a phase space of \( 2F(=6N) \) dimensions, \( S := k_B \cdot \log[\delta^2(p, q)/(2\pi \hbar)^F] \) with \( \delta^2(p, q) := \langle [W(p, q)^2 \delta p \delta q] \rangle \) is considered as an alternative to von Neumann’s entropy \( S := k_B \cdot \text{tr} \{ \log[\rho^{-1}] \} \).

1. Introduction

When Heisenberg first proposed his uncertainty relations [1] he had in mind an approximate equality rather than an inequality [2]. What he has written is

\[
\Delta p \cdot \Delta q \approx \hbar, \quad \Delta E \cdot \Delta t \approx \hbar
\]

(1)

with \( p := p_x, q := q_x \). But what is the (rough) meaning of \( \Delta x \) in all these cases of the variable \( x \)? What Heisenberg envisioned was that the length \( \Delta x := x' - x'' \) of an interval \([x' | x'']\) such that \( x \in [x' | x''] \) in a well qualified majority of cases (with a chance of about 80%, say). The product of such indeterminacies for a pair of canonically conjugate variables has to be at least nearly equal to Planck’s quantum of action \( \hbar \) in each natural state of affairs. This latter proviso stipulates that the state considered is not clouded by additional uncertainties of a purely subjective nature.

Later authors were not satisfied with this somewhat vague formulation, or with Heisenberg’s inductive argument. Very soon the text books rendered the well known inequality

\[
\sigma(p) \cdot \sigma(q) \geq \frac{1}{2} \hbar
\]

(2)

where \( \hbar = h/2\pi \) is the natural unit of action. Here

\[
\sigma(x) := \sqrt{\langle x^2 \rangle - \langle x \rangle^2}
\]

(3)

is the standard deviation (or ‘dispersion’) of the observable \( \hat{x} \), where \( v(x) := \mu ((x - \mu(x))^2) \) is the variance and \( \mu(x) \) the mean of \( \hat{x} \). The inequality (2) may be readily deduced from the commutation relation \( i[p, q] = \hbar \). Unfortunately, a similar statement for the pair \( (E, t) \) is not so easily obtained, \( t \) being a c-number so that \( i[E, t] = 0 \).

In addition, from Heisenberg’s original point of view the quantity of (2) is often too small by a factor of at least 10, as can be seen from the Table and Figures 1–3. This is easily taken care of [3] by using the spread

\[
\delta x := \sqrt{4 \pi \sigma(x)}
\]

(4)

instead of the dispersion \( \sigma(x) \). Yet in some (not so rare) cases the left hand side of the inequality (2) is much too large, sometimes even by an infinite factor.

<table>
<thead>
<tr>
<th>Distribution</th>
<th>( p(x) )</th>
<th>( \sigma(x) )</th>
<th>( \delta x )</th>
<th>( \delta [x] )</th>
</tr>
</thead>
<tbody>
<tr>
<td>Rectangular</td>
<td>( r_x(x) := \frac{a}{2\pi} \cdot \cos \left( \frac{a}{2\pi} x^2 \right) )</td>
<td>( \frac{\sqrt{a}}{12} )</td>
<td>( \frac{\sqrt{a}}{3} )</td>
<td>( a )</td>
</tr>
<tr>
<td>Lorentzian</td>
<td>( l_x(x) := \frac{a}{2\pi} \cdot \cos \left( \frac{a}{2\pi} x^2 \right) )</td>
<td>( \infty )</td>
<td>( \infty )</td>
<td>( 2\pi a )</td>
</tr>
<tr>
<td>Gaussian</td>
<td>( g_x(x) := \sqrt{\frac{2}{\pi} \cdot \frac{a}{2\pi} \cdot e^{-\frac{a}{2\pi} x^2}} )</td>
<td>( \frac{\sqrt{a}}{4\pi} )</td>
<td>( \frac{a}{a} )</td>
<td>( a )</td>
</tr>
<tr>
<td>Widedly split</td>
<td>( s_a (x) := \frac{a}{2} g_x (x + A) + \frac{a}{2} g_x (x - A) )</td>
<td>( \approx A )</td>
<td>( \approx \sqrt{4\pi A} )</td>
<td>( \approx 2a )</td>
</tr>
</tbody>
</table>

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Reprint requests to Prof. Dr. G. Süssmann, Garching.
Fig. 1. The width measures $\sigma(x)$, $\delta x$ and $\delta[x]$ of a typical distribution.

Both of these difficulties can be remedied by the following introduction of an uncertainty length:

$$\delta[x] := \frac{1}{\int p(x)^2 \, dx}, \quad (5)$$

where $p(x)$ denotes the probability density\(^1\) of the stochastic variable $\hat{x}$. Thus $\delta[x]$ may be interpreted as the (total) length of (all the) interval(s) that produce rectangle(s) of (total) area $h_x \cdot \delta[x] = 1$ ($=100\%$), the height $h_x$ being the mean value of the normalized $p(x)$ with $p(x)$ itself as its own weight factor.

\(^1\) The nominator 1 may be replaced by the square of the normalization integral $\int p(x) \, dx$, thus yielding a formulation which is formally emancipated from the normalization condition $\int p(x) \, dx = 1$. 

Fig. 2. A freely moving particle sequeezes its state spontaneously, whereby the true uncertainty area is conserved.

Fig. 3. Wigner's quasi-distribution $W(p, q)$ for a state $\psi(x) := [\phi(x + A) + \phi(x - A)]/\sqrt{2} + \cdots$ consisting of two amply separated branches.
2. Uncertainty Area

The inequality (2) suffers from inadequacies of another kind in situations where \( p \) and \( q \) are highly correlated. Typical examples are: (a) a particle after a long free motion [4]; (b) a strongly squeezed state of an oscillator [5]. Under such circumstances the two-dimensional phase space that is effectively claimed will resemble a strongly slanted parallelogram rather than a rectangle. Then, the product \( \delta p \cdot \delta q \) will be much larger than some more accurate measure \( \delta [p, q] \) of the genuinely inhabited portion of the phase plane.

Can this last idea be made precise? I think yes, having in mind a two-dimensional analog of the one-dimensional picture underlying (5). As \( p \) and \( q \) do not commute, I must resort to Wigner’s quasi-distribution \( W(p, q) \). This then yields the notion of the uncertainty area

\[
\delta [p, q] := \frac{1}{\int W(p, q)^2 \, dp \, dq},
\]

where the dominator surely is a positive quantity, Wigner’s \( W \) being real. Physical situations where \( W \) attains negative values are typically those with pronounced quantum illocalities, like a particle some time after passing through a beam splitter, as shown in Figure 3. The well-defined quantity \( \delta [p, q] \) may serve as an illuminating concept, especially in those extreme cases where the usual \( \delta p \cdot \delta q \) gives much too large of an estimate.

Such a misfortune can never happen here, because the uncertainty equation

\[
\delta [p, q] = 2 \pi \hbar
\]

turns out to hold true for each and every pure state \( \hat{\varrho} = | \psi \rangle \langle \psi | \). Here, it is \( \varrho(q', q'') := \psi(q') \psi(q'')^* \) which enters

\[
W(p, q) := \frac{1}{2 \pi \hbar} \int \varrho \left(q + \frac{1}{2} \hat{q}, q - \frac{1}{2} \hat{q} \right) e^{i p \varrho / \hbar} \, dq.
\]

A physical motivation for this definition [7] including the factors \( (2 \pi \hbar)^{-1} \) and \( \pm \frac{1}{2} \) is given in [8]. The easiest way to prove (7) is by using the identity [7]

\[
\int x(p, q) W(p, q) \, dp \, dq = \trc(\hat{x} \varrho).
\]

We need only to substitute \( 2 \pi \hbar W(p, q) \) for \( x(p, q) \) and, correspondingly, \( \hat{\varrho} = | \psi \rangle \langle \psi | \) for \( \hat{x} \), thus obtaining \( \delta [p, q] = 2 \pi \hbar / \trc \hat{\varrho}^2 = 2 \pi \hbar / \trc \hat{\varrho} = 2 \pi \hbar \).

For an arbitrary state, not necessarily a pure one, we have more generally \( 0 \leq \mathcal{G}^2 \leq \hat{\varrho} = \hat{\varrho}^* \leq I \), hence \( \delta [p, q] \geq 2 \pi \hbar \).

3. Entropy

This view is supported by a comparison with related concepts of quantum statistical thermodynamics. To this end if we generalize from one to \( F \) degrees of freedom (where \( F = 3N \)), and from pure states to mixed ones. Then with \( x := (x_1, \ldots, x_F) \) for \( x \in \{p, q\} \) we have

\[
\delta^F [p, q] \geq (2 \pi \hbar)^F.
\]

This squares well with the familiar thermodynamical fact that each microstate occupies the phase volume \( (2 \pi \hbar)^F \) in the mean. Thus we may consider

\[
G := \delta^F [p, q] / h^F \quad \text{with} \quad \delta^F [p, q] := h^F / \trc \hat{\varrho}^2 \quad \text{(11)}
\]

as the mean statistical weight of the thermodynamical state \( \hat{\varrho} \) under consideration. For macroscopic systems, even rather small ones, \( G \) has an exorbitant magnitude of about \( 10^{10^9} \), or more. On the other hand, Nernst’s Third Law of Thermodynamics states that at zero temperature the limiting value \( S = 0 \) obtains, corresponding to \( G = 1 \), which means that the ground state is essentially nondegenerate.

The natural logarithm \( \log := \log_e \) of the phase volume \( \delta^F [p, q] \), measured in its natural units \( (2 \pi \hbar)^F \), is a statistical counterpart of the entropy notion [9] in units of Boltzmann’s constant \( k_B \). Thus we may propose

\[
S = k_B \cdot \log G \quad \text{(12)}
\]

= bit \cdot \log_e G \geq 0 as an alternative closely related to the well-known definition \( S := - k_B \trc (\hat{\varrho} \log \hat{\varrho}) \), originally proposed by Boltzmann, Gibbs, and von Neumann (and later generalized into informatics by Szilard and Shannon). That \( S \approx \tilde{S} \) is true for typical macroscopic states can be checked by the prototypical case of a simple yes–no distribution: \( G \) microscopic states of equal weight have

\[
S / k_B = - \log \Sigma_{\varrho=1}^G p^2 = - \log (G G^{-2})
\]

\[
= \log G = - G^{-1} \log (G^{-1})
\]

\[
= - \Sigma_{\varrho=1}^G p \log p = \tilde{S} / k_B.
\]

It has at most a microscopic multiplicity \( G \ll F! \) where \( F \geq 10^{12} \).
The microcanonical ensemble, from which Gibbs has derived the canonical one, is a case in point.

4. Conclusion

Returning to the phase plane $(p, q)$, we may summarize by stating that each pure quantum state is a minimum uncertainty state according to a rather natural definition $\delta[p, q]$ of the combined $(p, q)$-uncertainty. Mixed states have larger uncertainties, their statistical weight $G := \delta[p, q]/2\pi\hbar$ being an appropriate measure of impurity.

Acknowledgements

In discussions with Wolfgang Schleich [10] I have realized that my former considerations about $\delta p, \delta q, \delta[p, q]$ etc. might be useful enough to be published. This fine workshop is a good occasion. It is a pleasure to thank the quantum optics colleagues around Garching and Albuquerque, especially Marlan Scully, for all the hospitality and the agreeable atmosphere of scientific exchange. I owe Berthold-Georg Englert my thanks for advising me on the English text.