The “Triptych” Function – Determining the Basins of Attraction of One-Dimensional Maps

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We show that a modification of the path length method, introduced by R. Richter et al. [Z. Naturforsch. 49 a, 871 (1994)], gives us a method for finding basins of attraction of one-dimensional maps with multiple attractors.

This function measures, therefore, the average increment in the graph \( X_{i+k} \) versus \( X_i \). If the trajectory is a periodic orbit with period \( k \), the function \( T_k[F_n] \) will tend to zero, when \( N \to \infty \). If the trajectory has a different period, this function will give us the average increment at each step. Therefore, by finding the value of \( T_k \) as a function of the initial condition (for \( N \to \infty \)) we can distinguish the basins of attraction of different attractors. For instance, if a map has two competing attractors, in the interval [0, 1], the first one with period 1 and the second with period 2, the average triptych function of first order will give zero in the basin of attraction of the first attractor and a value different from zero (the increment between the points on the attractor with period 2) in the basin of attraction of the second attractor. This fact gives us a quick method for identifying the different domains of attraction. We note that, as an interesting feature of this procedure, this function can distinguish also between two attractors with the same period, except if the increment, at each step, is equal for the two attractors.

Let us consider two examples. The Tent-Bernoulli (TB)-map has the form

\[
X_{i+1} = a X_i, \quad \text{if } 0 \leq X_i < 1/2,
\]

\[
= 2a X_i - 1, \quad \text{if } 1/2 \leq X_i < 1.
\]

We take \( b \in [0, 1] \) and we restricted, for our convenience, the domain of parameter \( a \) to be the interval [0.9, 1]. The parameter \( b \) is a relative measure of the gap of map (1) at the point \( X = 1/2 \) and \( a \) is a measure of the maximum height of map (1). This map is a discontinuous one-dimensional map which can be seen as a transition between the tent map and the Bernoulli shift: if \( a = 1 \) and \( b = 0 \) we get the tent map and, if \( a = 1 \) and \( b = 1 \), the Bernoulli shift. Several analytical results can be obtained for this map [5]. For

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Fig. 1. $X_n$ versus $b$, for the TB-map, with the initial condition $X_0 = 0.3$ and $a = 1$. 

Fig. 2. $X_n$ versus $b$, for the TB-map, with the initial condition $X_0 = 0.45$ and $a = 1$. 

Fig. 3. Trintychn function of first-order for the TB-map in the interval 

Fig. 4. Basin of attraction of the period 1 attractor (black) in the $b-X_n$.
Fig. 5. The triptych function has different values for two attractors with the same period for the map (2).

Fig. 6. Basins of attraction of a period 1 attractor and of another one with period 4 for the map (2).

Fig. 7. Basins of attraction of a chaotic attractor and of a period 1 attractor for the map (2).

Fig. 8. Basins of attraction of a period 1 attractor (white) and of a period 2 attractor (black) for the map (2), with $a = 0.9$. 

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a fixed $a$, if $b$ is varied from 0 to 1 the behavior of the map changes between chaotic and regular. We are interested here in the domains of $a$ and $b$ where two different attractors coexist. The analysis of the first and second return maps shows that, in the interval $b \in [0, 1/2]$, one attractor with period 1 (P-I) exists if $1/2 - (1/4a) < b < 1 - (1/2a)$ and one attractor of period 2 (P-II) exists if $b > 1/2 - (1/8a^2)$. Therefore, the coexistence of these attractors occurs if $1/2 - (1/8a^2) < b < 1 - (1/2a)$ [see Figs. 1 and 2].

The basins of attraction of these two periodic attractors, for $b = 0.45$ and $a = 1.0$, obtained by using the first-order triptych function are represented in Figure 3. The value of the triptych function for the basin of attraction of P-I is zero. We note that, near $X = 0$, the basins of attraction are interwoven. In Fig. 4 the entire basin of attraction of P-I in the $b - X_0$ space is shown. The number of iterations is $N = 2000$ for all figures.

The method was applied also for determining the basins of attraction of a family of continuous one-dimensional maps with the form [6]

$$X_{i+1} = b X_i + (-7b + 135a/4) X_i^2 + (15b - 189a/2) X_i^3 + (243a/4 - 9b) X_i^4.$$ (2)

This map has in the interval $[0, 1]$, several coexisting (periodic or chaotic) attractors with blended basins of attraction for several values of the parameters $a$ and $b$. We have constructed this map by violating the conditions of Singer’s theorem [7] for the existence of only one attractor for continuous one-dimensional maps and, by imposing some additional conditions for getting a simple map, with two parameters, in the interval $[0, 1]$. For instance, Fig. 5 shows the basin of attraction for two attractors both with period 2, if $a = 0.94$ and $b = 8.10$. In Fig. 6, the triptych function permits us to determine the basin of an attractor with period 4 and the basin of another attractor with period 1. In Fig. 7, where we used the triptych function of first-order, we can see the basins of attraction of a chaotic attractor and of a period 1 attractor. Figure 8 shows the entire basin of attraction of two attractors in the interval $b \in [7.1, 7.3]$, if $a = 0.9$.

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