Noisy Maps near Crises*
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We address the escape rate for single-humped maps near a boundary crisis multiplicatively coupled to weak uncorrelated noise. A scaling law for the rate is derived predicting a stabilization of deterministic transient chaos by noise. Generalizations to maps near interior crises and band merging points are given.

A large variety of problems arising in very different scientific areas can be approximately described by low dimensional nonlinear dynamics [1]. Of particular interest are sudden qualitative changes that can occur in such systems, for instance bifurcations or crises [2]. Typically, in such a low dimensional description small perturbations arising from a large number of fast variables are neglected. However, the influence of this “environment” should be included in a more realistic model in the form of weak noise, in particular close to the above-mentioned sudden qualitative changes of the deterministic dynamics.

We consider a one-dimensional dynamics in discrete time $n$ in the presence of weak multiplicative noise:

$$x_{n+1} = f(x_n) + \sigma g(x_n) \xi_n.$$  \hspace{1cm} (1)

Here, $f(x)$ is a single-humped map of the real axis with a maximum of order $z > 0$ at $x = x^*$,

$$f(x) = 1 + A - b|x-x^*|^z + o(|x-x^*|^z),$$ \hspace{1cm} (2)

where $A$ is a small parameter, $-1 < A < 1$, and $b > 0$. Further, $f(x)$ is strictly monotonical on both sides of the maximum $x^*$ and continuously differentiable everywhere with the exception of $x = x^*$ for $z \leq 1$. The $x$-scale is chosen such that $f(x)$ has an unstable fixed point at $x = 0$, $f(0) = 0$, and a second zero at $x = 1$, $f(1) = 0$, implying $f'(0) > 1$, $f'(1) < 0$, and $0 < x^* < 1$. Thus, at $A = 0$ the map undergoes a boundary crisis [2] and shows fully developed chaos [3]. A well known example with $z = 2$ is the logistic map

$$f(x) = 4(1 + A)x(1-x).$$ \hspace{1cm} (3)


The noise-strength $\sigma$ in (1) is required to be small, $0 < \sigma \ll 1$, and the noise-coupling function $g(x)$ to be bounded on $\mathbb{R}$ and continuous at $0$, $x^*$, and 1. The noise $\xi_n$ is given by independent, identically distributed random numbers with a probability distribution $P(\xi)$ that decreases faster than $1/|\xi|^{1+1/z}$ for large $|\xi|$. In order to determine the probability density of the stochastic process (1) in the quasi-stationary state we developed a novel perturbation method [4] in the two small parameters $A$ and $\sigma$ about the stationary probability density $g(x)$ at fully developed chaos $A = 0$ [3] in the absence of noise $\sigma = 0$. This method has certain similarities to singular perturbation theory for differential equations. For $z > 1$ and $A < 0$ the deterministic map $f(x)$ shows periodic windows [1] giving rise to considerable difficulties for any perturbation theory about $A = \sigma = 0$ [5]. We succeeded to derive the necessary and sufficient condition

$$\sigma \gg |A|^{z-1} \quad (z > 1, A < 0)$$ \hspace{1cm} (4)

for the validity of our method [4]. From the quasi-stationary probability density one finds the escape rate $k$ out of the unit interval $[0,1]$.

$$k = g(x^*) \left( \frac{\sigma}{b} \right)^{1/z} F \left( \frac{A}{\sigma} \right), \quad F(x) = 2 \int_0^1 dy y^{1/z} h_{x*}(y - x).$$ \hspace{1cm} (5)

The function $h_{x*}(x)$ is given as the $l \to \infty$ limit of

$$h_l(x) = \frac{\pi}{2} e^{i k x} \left( \prod_{m=0}^{\infty} \tilde{P}(T_m k) \right) T_m = \frac{g(f_{x*0}(x))}{d f_{x*0}(x)}$$ \hspace{1cm} (6)

where $\tilde{P}(k) = \frac{1}{2\pi} \int_{-\infty}^{\infty} \xi e^{-ik \xi} P(\xi)$ is the Fourier transform of the noise distribution. For the single-humped maps at the boundary crisis $f_{x*0}(x)$ considered here one has $T_0 = g(x^*)$, $T_1 = g(1)/f'(1)$, $T_m = g(0)[f'(1)f'(0)^{m-1}]$, $m \geq 2$.  

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Our central rate formula (5) has the form of a scaling law [6, 7]. It becomes asymptotically exact for small $\Delta$ and $\sigma$ respecting the extra condition (4) and indeed compares very well with the numerical simulations shown in Figure 1. Apart from $g(x^*)$, which is indeed comparable with the numerical simulations of (1) for $\Delta = 10^{-5}$ (circles), $\Delta = 10^{-3}$ (crosses), and $\Delta = 10^{-7}$ (triangles). The statistical uncertainty is about 1%. For $\Delta = 10^{-7}$ the numerical results are systematically slightly above the theoretical line which clearly is a finite $\Delta$ and $\sigma$ effect.

Fig. 1. Escape rate $k$ versus noise-strength $\sigma$ for the logistic map (3) with additive Gaussian noise $g(x) \equiv 1$, $P(\xi) = (2\pi)^{-1/2} e^{-\xi^2/2}$. The solid line is the theoretical prediction (5), (6) with $g(x^*) = 2/\pi$ [1]. The symbols are results from numerical simulations of (1) for $\Delta = 10^{-5}$ (circles), $\Delta = 10^{-3}$ (crosses), and $\Delta = 10^{-7}$ (triangles). The statistical uncertainty is about 1%. For $\Delta = 10^{-7}$ the numerical results are systematically slightly above the theoretical line which clearly is a finite $\Delta$ and $\sigma$ effect.

The escape rate $k$ can be written under the form $k = 2g(x^*)(\Delta/b)^{1/2} G(\sigma/\Delta)$, where the new scaling function $G(x)$ is related in an obvious way to $F(x)$. Closer inspection shows that on $\mathbb{R}_+$ the function $G(x)$ exhibits a global maximum at $x_{\text{max}} > 0$ whenever the noise distribution is symmetric, $P(-\xi) = P(\xi)$, and $z > 1$. As a function of $z$, this minimum $x_{\text{min}}$ is monotonically increasing, becoming $0$ for $z \to 1$ and proportional to $z$ for large $z$. The minimum value $G(x_{\text{min}})$ monotonically decreases from $1$ for $z = 1$ to $1/2$ for $z \to \infty$. Thus, for any $\Delta > 0$ sufficiently small noise-strengths $\sigma$ will lead to smaller rates $k$ than in the absence of noise $\sigma = 0$, see Figure 1. In other words, the noise induces a stabilization of deterministic transient chaos [8]. The same effect occurs for sufficiently large $z$-values even if the noise distribution is no longer symmetric. A simple intuitive explanation is possible only in the particular case that all the $T_m$ in (6) vanish for $m \geq 1$ [4].

As a particular example we first consider the symmetric Lévy distributions given by $P(\xi) = \exp \{-|\xi|^z\}$, $0 < \mu \leq 2$. For $\mu = 2$ and $\mu = 1$ one recovers Gaussian and Lorentz distributions ($P(\xi) = \pi(1 + \xi^2)^{-1}$), respectively. From (6) one readily finds that $h_{\infty}(x) = P(x/U_\mu)/U_\mu$, where $U_\mu = \left(\sum_{m=0}^{\infty} T_m|\mu|^{1/\mu}\right)^{1/\mu}$. Consequently, the scaling law (5) is equivalent to

$$
\frac{r}{2} \frac{A_\infty}{T_m} = \frac{A_{\infty}^2}{T_m} - T_m^\mu \quad \text{for } \alpha = 1, \quad (8)
$$

where

$$
A_{\infty} = \max_{m \geq 0} T_m \quad \text{for } 0 < \alpha \leq 1,
A_{\infty} = \left(\sum_{m=0}^{\infty} T_m^{-1}\right)^{-1/\alpha} \quad \text{for } \alpha > 1,
$$

and $\beta(x) \to (\alpha - 1)(2 - \alpha)(4 \ln f'(0))$ for $x \to \infty$. From (7)–(9) one readily finds that the rate (5) is dominated by an Arrhenius-like factor $\exp \{-|\Delta/\sigma A_\infty|^z\}$ in the deep precritical regime $\Delta \ll -\sigma$ (but still respecting (4)), in agreement with [9, 10]. The pre-exponential factor of the rate is algebraic in $|\Delta|$ and $\sigma$ for $\alpha \leq 1$ and $\alpha = 2$, whereas for the remaining $\alpha$-values the $|\Delta|$ and $\sigma$ dependence is stronger than any power law, similarly as in (9).

Our method to determine the quasi-stationary probability density and the rate can be generalized to single-humped maps near an interior crisis [2] or...
a band merging point [1]. In this case, a suitable iterate of the map, say \( f^p(x) \), exhibits a boundary crisis when restricted to any of \( p \) properly chosen disjoint subintervals \( I_i, i = 1, \ldots, p \), of \([0,1]\). Equation (2) generalizes to \( f(x) = x_1 + \Delta - b |x - x^*|^r + \ldots \), where \( x_1 \) is mapped under \( f^p(x) \) on the unstable periodic orbit which collides with the strange attractor at the crisis or band merging point. At \( \Delta = 0 \), the intervals \( I_i \) are mapped onto each other by \( f(x) \) and onto themselves by \( f^p(x) \). If one denotes by \( p \cdot k \) the escape rate in the quasi-stationary state from \( I_i \) after \( p \) time steps, then exactly the same result (5), (6) for \( k \) is found, independent of the particular interval \( I_i \) [4]. Further generalizations to multiple-humped maps and exponentially correlated Gaussian noise are possible [4].

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