Exact Solutions of Two Body Dirac Equations

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A set of exact eigen values and eigen functions for the two body Dirac problem described by the Hamiltonian

\[ H = (\tilde{\alpha}_1 - \tilde{\alpha}_2)\tilde{p} + \beta_1 m_1 + \beta_2 m_2 + \frac{1}{2}(\beta_1 + \beta_2)\lambda r \]

are obtained using the properties of Supersymmetric Quantum Mechanics.

Recently the two body Dirac equation has been studied by several authors [1]–[4] as it is the natural way to study a two fermion system and has immediate application in particle physics, particularly in the study of Regge trajectory and light meson spectra.

Very recently, Semay and Ceuleneer [4], [5] have studied a two body Hamiltonian given in the centre of mass frame by (units are such that \( \hbar = c = 1 \))

\[ H = (\tilde{\alpha}_1 - \tilde{\alpha}_2)\tilde{p} + \beta_1 m_1 + \beta_2 m_2 + \frac{1}{2}(\beta_1 + \beta_2)\lambda r, \quad \text{(1)} \]

where \( \tilde{p} = -i\vec{\nabla} \) and \( \tilde{r} = \tilde{r}_1 - \tilde{r}_2. \quad \text{(2)} \]

The eigen states of (1) will be 16 component spinors [2], [4]. For central diagonal potentials they can be reduced to simple forms so that one needs only to solve a second order eigenvalue problem involving the radial function \( \varphi(r) \) only.

In this note we shall derive a set of exact eigenvalues and eigen functions for the radial wave function problem when \( \lambda \), the coupling constant, satisfies certain constraint relation. For \( l = 0 \), we shall verify our results by solving the eigen value problem exactly for all values of \( \lambda \). Our method is based on the properties of supersymmetric quantum mechanics [6]–[9] which has been used before to calculate exact eigen values [10]–[13] of Schrödinger problems.

The spinor eigen states can be written as [5]

\[ r\Psi/N_0 = \frac{E + M + \lambda r}{2E} \varphi |1; J 1 J J_z\rangle - \frac{1}{E - M'} \varphi' |2; J 1 J J_z\rangle + \frac{\sqrt{J(J + 1)} \varphi}{E - M^2} |2; 0 0 J J_z\rangle - \frac{1}{E - M'} \varphi' |3; J 1 J J_z\rangle + \frac{\sqrt{J(J + 1)} \varphi}{E + M^2} |3; 0 0 J J_z\rangle - \frac{E - M - \lambda r}{2E} \varphi |4; J 1 J J_z\rangle, \quad \text{(3)} \]

where \( N_0 \) is a normalization factor, \( |i; l J J_z\rangle \) are angular basis states and \( M \) and \( M' \) are defined as \( M = m_1 + m_2 \) and \( M' = m_1 - m_2 \). The natural parity \((l = J)\) radial equation for \( \varphi(r) \) is

\[ \varphi'' + \left[ \left( E^2 - (M + \lambda r)^2 \right) \left( E^2 - M'^2 \right) \right] \varphi = 0. \quad \text{(4)} \]

Before casting (4) in super symmetric form we give below a summary of the salient features of super symmetric quantum mechanics (SUSYQM) in one dimension. In one dimension the Hamiltonian of SUSYQM is given by

\[ H^S = \{ Q^+, Q \} = \begin{pmatrix} H_+ & 0 \\ 0 & H_- \end{pmatrix}, \quad \text{(5)} \]

where

\[ H_\pm = -\frac{1}{2}d^2/dx^2 + V_\pm(x), \quad \text{(6)} \]

\[ V_\pm = \frac{1}{2} (W^2(x) \pm dW(x)/dx). \quad \text{(7)} \]
$W(x)$ is called the superpotential and $Q$, $Q^+$ the supercharges whose explicit forms are
\begin{align}
Q &= \frac{1}{\sqrt{2}}(p - iW) \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}, \\
Q^+ &= \frac{1}{\sqrt{2}}(p + iW) \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}.
\end{align}
(8)

The relations obeyed by $Q$, $Q^+$, and $H^s$ are the following:
\begin{align}
[H^s, Q] &= [H^s, Q^+] = 0, \\
Q^2 &= Q^+ = 0.
\end{align}

The eigen states of $H^s$ are
\begin{equation}
\varphi^n(x) = \begin{pmatrix} \varphi^+_n(x) \\ \varphi^-_n(x) \end{pmatrix}.
\end{equation}
(10)

If supersymmetry is unbroken, the ground-state energy is zero and the ground-state wave functions are of the form
\begin{equation}
\varphi^n(x) = \begin{pmatrix} \varphi^+_n(x) \\ 0 \end{pmatrix}
\end{equation}
(11)
depending on the normalisability of $\varphi^+_n(x)$ or $\varphi^-_n(x)$. Now if $|\Psi\rangle$ is the ground state then
\begin{equation}
Q|\Psi\rangle = Q^+|\Psi\rangle = 0.
\end{equation}
(12)

From (8) and (9) it follows that
\begin{equation}
\varphi^0_{\pm}(x) = \exp \left( \pm \int x W(t) dt \right).
\end{equation}
(13)

For (4) a suitable ansatz for $W$ is
\begin{equation}
W = ar + \frac{b}{r} + d + \sum^{n}_{i=1} \frac{g_i}{1 + gi r}.
\end{equation}
(14)

We now write (4) as
\begin{equation}
\varphi'' - (V^+(r) - E^+) \varphi = 0,
\end{equation}
(15)

where
\begin{equation}
V^+(r) = W^2 + W'
\end{equation}
and $E^+$ is the corresponding eigenvalue for $W^2 + W'$ (bosonic sector say). Comparing (15) and (4) and using (14) we get, (equating $W^2 + W' - E^+$ with the term within brackets in (4)),

\begin{align}
b &= l + 1 \\
a &= -\lambda \left(1 - \frac{M'^2}{E^2}\right)^{3/2} \\
d &= \frac{M\lambda}{4a} \left(1 - \frac{M'^2}{E^2}\right) = -\sum^{n}_{i=1} g_i, \\
2ab + a + 2an + d^2 - E^+ &= \frac{1}{4} \left(M^2 + M'^2 - E^2 - \frac{M^2 M'^2}{E^2}\right),
\end{align}
(16)

and
\begin{equation}
2dg_i - 2bg_i^2 - 2a + 2\sum^{n}_{i \neq j} \frac{g_i^2 g_j}{g_i - g_j} = 0,
\end{equation}
(16e)

When $n \leq 2$ the above equations can be solved analytically. We consider the following cases (we assume that $m_1$ and $m_2$ are not simultaneously zero).

(i) $n = 1$.

It can be easily seen from (16e) that
\begin{equation}
g_1^2 = -\frac{a}{b + 1}.
\end{equation}
(17)

The other parameters can be found by solving (16a) - (16c). Eliminating $a$, $d$, and $b$ from (16d) and taking $E^+ = 0$ (it corresponds to the ground state of the SUSY potential) we get
\begin{equation}
E^2 = M^2(2l + 5)(l + 2),
\end{equation}
(18)

where we have neglected the solution $E^2 = M'^2$. Further, $\lambda$ satisfies the relation
\begin{equation}
\lambda = M^2 \frac{l + 2}{2} \left(1 - \frac{M'^2}{M^2(2l + 5)(l + 2)}\right)^{1/2}.
\end{equation}
(19)

The wave function is given by
\begin{equation}
\varphi(r) = r^{l+1}(1 + g_1 r)e^{-ar^2/2 + dr},
\end{equation}
(20)

where
\begin{equation}
d = -g_1 = \frac{a}{l + 2},
\end{equation}
(20a)

and
\begin{equation}
a = -\lambda \left(1 - \frac{M'^2}{E^2}\right)^{3/2}.
\end{equation}
(20b)

(ii) $n = 2$. 


Here we first solve $g_1, g_2$ using (16e). In fact one solves for $g_1 g_2$ and $g_1 g_2$ and gets

$$g_1 + g_2 = \sqrt{-\frac{a (4l + 9)}{(l + 2)(l + 3)}}$$

and

$$g_1 g_2 = -\frac{a}{l + 3}.$$  

Then from (16b) and (16c) one gets

$$\lambda = \frac{M^2}{2} \sqrt{1 - \frac{M^2}{E^2} \frac{(l + 2)(l + 3)}{4l + 9}}$$

and

$$a = -\frac{M^2}{4} \left(1 - \frac{M^2}{E^2}\right) \frac{(l + 2)(l + 3)}{4l + 9},$$

and $E$ is given by

$$E^2 = M^2 \left[3 + \frac{(2l + 7)(l + 2)(l + 3)}{4l + 9}\right].$$

To check our results for $l = 0$ we solve (4) putting

$$x = \frac{(M + \lambda r)^2}{\beta},$$

where $\beta$ is given by

$$\beta = \frac{2E\lambda}{\sqrt{E^2 - M^2}}.$$  

Equation (4) reduces to (for $l = J = 0$)

$$x\Psi'' + \left(\frac{1}{2} - x\right)\Psi - \bar{a}\Psi = 0,$$

where

$$\Psi(x) = e^{x/2}\varphi(x)$$

and

$$\bar{a} = -\frac{E(E^2 - M^2)^{1/2}}{8\lambda} + \frac{1}{4}.$$  

The general solution of (25) is given [14] by

$$\varphi = AM \left(\bar{a}, \frac{1}{2}, x\right) + BM \left(\bar{a} + \frac{1}{2}, \frac{3}{2}, x\right),$$

where $A, B$ are normalization factors and $M(a, \beta, x)$ is the confluent hypergeometric function.

$M (\bar{a}, \frac{1}{2}, x)$ and $M (\bar{a} + \frac{1}{2}, \frac{3}{2}, x)$ are respectively the even and odd parity solutions (the parity refers to the variable $x$, not $r$). For $N = 1$ we take the even parity solution. The eigenvalues are given by the roots of $M (\bar{a}, \frac{1}{2}, x_0)$ where $x_0 = x(r = 0)$ i.e. $M (\bar{a}, \frac{1}{2}, x_0) = 0$, when

$$x_0 = M^2/\beta.$$  

To check our exact results with the analytic solutions (26) (for $l = 0$) let us assume that $\lambda$ takes a value as to make $\bar{a} = -1$. Then $M (\bar{a}, \frac{1}{2}, x_0) = 0$ gives

$$1 - 2x_0 = 0$$

or

$$x_0 = \frac{M^2}{\beta} = \frac{1}{2}.$$  

From the definition of $\bar{a}$ and $x_0$ we get two simultaneous equations for $\lambda$ and $E$, viz.

$$E(E^2 - M^2)^{1/2} = 10\lambda$$

and

$$M^2 = \frac{E\lambda}{\sqrt{E^2 - M^2}}.$$  

Solving for $E$ and $\lambda$ we get

$$E^2 = 10M^2$$

and

$$\lambda = M^2 \left(1 - \frac{M^2}{10M^2}\right)^{1/2},$$

which are the same as those given by (18) and (19), respectively, provided we take $l = 0$.

For $n = 0$, we take the odd parity solution, viz. $M (\bar{a} + \frac{1}{2}, \frac{3}{2}, x)$. If one takes $\bar{a} + \frac{1}{2} = -1$ and $M (\bar{a} + \frac{1}{2}, \frac{3}{2}, x_0) = 0$ (which gives $x_0 = \frac{3}{2}$) one again gets two simultaneous equations for $E$ and $\lambda$. Solving them one gets $E$ and $\lambda$ given by (22) and (21c), respectively, provided one puts $Z = 0$ in these equations.

For the particular case $m_1 = m_2 = 0$ (which means $M = M' = 0$) the potential becomes a function of $r^2$ only, and hence $n$ in (14) can only have even values. The super potential then should have the terms

$$\sum_{i=1}^{n} \frac{g_i}{1 + g_i r} \text{ replaced by } \sum_{i=1}^{n} \frac{2g_i r}{1 + g_i r^2}.$$
For example, if one takes $n = 2v$ and $M = M' = 0$, $E^+ = 0$ in (16d) one gets, using (16b) and (16c)

$$E = \sqrt{2\lambda(4v + 2l + 3)},$$

which is identical with the result obtained in [4] and [5].

To conclude, we have used the properties of SUSYQM to obtain two sets of exact eigenvalues and eigen functions for the Hamiltonian given by (1) when $\lambda$ satisfies a constraint relation. These solutions would act as benchmarks against which the accuracy of numerical and analytical solutions can be judged.

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