Some Monotonicity and Convexity Results for Integral Means

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Analogues and extensions of the classical result on monotonicity of $L^p$ norms and log convexity of $p^\infty$ power of $L^*$ norms are established.

In the discussion of criteria for magnetohydrodynamic stability one encounters inequalities of the type

$$\left\{ \frac{B^{-1} dl}{B^{-3} dl} \right\} \frac{B^{-1} dl}{B^{-3} dl} \geq \left\{ \frac{B^{-2} dl}{B^{-3} dl} \right\}^2,$$  \hspace{1cm} (1a)

$$\left\{ \frac{B^{-1} dl}{B^{-4} dl} \right\} \frac{B^{-1} dl}{B^{-3} dl} \geq \left\{ \frac{B^{-2} dl}{B^{-3} dl} \right\} \frac{B^{-3} dl}{B^{-3} dl},$$  \hspace{1cm} (1b)

where $l$ is the arc length, $B(l)$ the (non-vanishing) magnetic field, and $\frac{\ldots}{\ldots} dl$ the integral over a closed line of force. While (1a) is of the Schwarz type, (1 b) is not. Inequalities (1) can be reduced to monotonicity of the functional

$$\left\{ \int_0^1 f^2(x) \, dx \right\} \int_0^1 f^{-2}(x) \, dx, \quad 0 < f < \infty$$

in the real parameter $x$ with a positive weight function $\mu(x)$.

The aim here is to establish certain integral inequalities which are in the spirit of the following classical result on $L^p$ norms (see, for instance, [1]):

Given a measure space $(\Omega, \mathscr{A}, \mu)$, where $\mu$ is a probability measure, and given $f$ a non-negative real-valued function on $\Omega$, the functional $I(p) = \int f^p \, d\mu$, defined for $p \in R$, is log convex in $p$.

The log convexity result in Theorem 1 is an extension of [1] to the case where, instead of $L^p$ norms, a more restrictive (though simpler) condition than (1) is imposed, i.e.

$$\text{Condition } \gamma: \begin{cases} f_i(x) \geq f_j(y) \Rightarrow f_i(x) \geq f_j(y), & j \neq i, \\ i,j = 1, \ldots, m, \quad \forall (x,y) \in A \times A \end{cases}$$

then all $m^2$ second partial derivatives of $I_A$ are non-negative.

Theorem 1. Let $(\Omega, \mathscr{A}, \mu)$ be a positive measure space and let $f: \Omega \rightarrow R_+$ be a measurable function. Then the functional $I_A(\alpha)$ for $\alpha \in R$ and $A \subset \Omega$ (see (2)) is monotonically increasing in $|\alpha|$ unless $f$ is constant.

First Proof: It suffices to consider the case $\alpha \geq 0$ since $I_A(-\alpha) = I_A(\alpha)$. Let us begin by observing that in order for $\int_A f^2 \, d\mu$ and $\int_A f^{-2} \, d\mu$ both to be finite, we must have $\mu(A) < \infty$. To see this, note that

$$\mu(A) = \mu\{ x: f(x) \geq 1 \} + \mu\{ x: \frac{1}{f(x)} > 1 \} \leq \int_A f^2 \, d\mu + \int_A f^{-2} \, d\mu.$$

By rescaling we may assume, without loss of generality, that $\mu(A) = 1$. If $\alpha < \beta$, we have, by applying Jensen's inequality $g(\int h \, d\mu) \leq \int g(h) \, d\mu$ with the con-
vex function \( g(u) = u^{\beta/\alpha} \),

\[
(I_A(x))^{\beta/\alpha} \leq \left( \int_A f^\beta \, d\mu \right) \left( \int_A f^{-\beta} \, d\mu \right) = I_A(\beta).
\]

It thus follows that

\[
I_A(x) \leq I_A(\beta^{\alpha/\beta}).
\]

It suffices to prove \( I_A(\beta) > 1 \) for every \( \beta \) (unless \( f \equiv \text{constant} \)). But this is easy for we have, by the Cauchy-Schwarz inequality,

\[
1 = \left( \int_A f^{\beta/2} f^{-\beta/2} d\mu \right) \left( \int_A f^{\beta/2} f^{-\beta/2} d\mu \right)^{1/2} \geq \left( \int_A f^\beta d\mu \right) \left( \int_A f^{-\beta} d\mu \right)^{1/2} = \sqrt{I_A(\beta)},
\]

with equality only if \( f^{\beta/2} \) is a multiple of \( f^{-\beta/2} \), i.e. only if \( f = c \).

**Example:** Taking \( \mu \) as counting measure, we have for \( \alpha_i > 0, i = 1, \ldots, N \) as an application of Theorem 1 the following result for sums:

\[
\left( \sum_{i=1}^N a_i^2 \right) \left( \sum_{i=1}^N a_i^{-2} \right)
\]

is a monotonically increasing function of \( |\alpha| \).

**Second proof:** Let us write \( I_A(x) \) as

\[
I_A(x) = \frac{1}{2} \left\{ \left( \int_A f^\alpha(x) \, d\mu(x) \right) \left( \int_A f^{-\alpha}(y) \, d\mu(y) \right) + \left( \int_A f^\alpha(y) \, d\mu(y) \right) \left( \int_A f^{-\alpha}(x) \, d\mu(x) \right) \right\}
\]

\[
= \frac{1}{2} \int_A \int_A \left\{ \left( f(x) \right)^\alpha \left( f(y) \right)^\alpha + \left( f(x) \right)^{-\alpha} \left( f(y) \right)^{-\alpha} \right\} d\mu(x) d\mu(y)
\]

and use the fact that

\[
\alpha \to \alpha^2 + \frac{1}{\alpha^2}, \quad \text{for} \quad \alpha > 0,
\]

is a monotonically increasing function of \( |\alpha| \) (unless obviously \( \alpha = 1 \)).

This second proof has the advantage of providing the following \( m \)-variable extension.

**Theorem 2.** The \((\Omega, \mathcal{A}, \mu)\) be a positive measure space and \(f_1, \ldots, f_m: \Omega \to \mathbb{R}^+\) be measurable functions, then 
\( I_A(\alpha_1, \ldots, \alpha_m) \) increases on each variable separately for every \( A \) in \( \mathcal{A} \) if, and only if, Condition (\( \delta \)) holds, i.e.

\[
\frac{\partial}{\partial \alpha_i} F_{x,y} \geq 0 \quad \text{for each variable} \quad \alpha_i,
\]

**Condition (\( \delta \)):**

\[
\forall (x, y) \in \Omega \times \Omega \text{ a.e. } d\mu(x) \otimes d\mu(y).
\]

where

\[
F_{x,y}(\alpha_1, \ldots, \alpha_m) = \prod_{i=1}^m \left( \frac{f(x)}{f_i(y)} \right)^{\alpha_i}
\]

and \( a_i = f_i(x)/f_i(y) \), we have

\[
\frac{\partial F}{\partial \alpha_i} = (a^2 - a^{-2}) \log a_i
\]

for \( i = 1, \ldots, m \). But using the definition of \( a_i \) and Condition (\( \gamma \)) we see that

\[
\log a_i \geq 0 \quad \iff \quad \frac{f_i(x)}{f_i(y)} \geq 1 \quad \iff \quad \frac{f_i(x)}{f_i(y)} \geq 1,
\]

\[
\forall j \neq i, j = 1, \ldots, m \Rightarrow a^2 - a^{-2} \geq 0,
\]

and similarly

\[
\log a_i \leq 0 \quad \Rightarrow \quad a^2 - a^{-2} \leq 0.
\]

If Condition (\( \gamma \)) holds, we can in fact say more, i.e. we have

**Theorem 3.** Let \((\Omega, \mathcal{A}, \mu)\) be a positive measure space and \(f_1, \ldots, f_m: \Omega \to \mathbb{R}^+\) be measurable functions, then 
\( I_A(\alpha_1, \ldots, \alpha_m) \) is a convex function of \( \alpha_i \). If, in addition, Condition (\( \gamma \)) is satisfied, then all \( m^2 \) second partial derivatives of \( I_A \) with respect to \( \alpha_i \) are non-negative.

**Proof:** We begin by calculating the second-order partials

\[
\frac{\partial^2 F}{\partial \alpha_i \partial \alpha_j} = \int \left( a^{2i} - a^{-2j} \right) \log a_i \log a_j d\mu.
\]

But

**Condition (\( \gamma \))** \( \Rightarrow \) \(( \log a_i ) ( \log a_j ) \geq 0 \).

We next establish the convexity. If we let

\[
c_i = \log a_i, \quad i = 1, \ldots, m,
\]

the positive semidefiniteness of the matrix \( \frac{\partial^2 F}{\partial \alpha_i \partial \alpha_j } \) is an immediate consequence of the positive semi-
definiteness of the matrix $D = \{d_{ij}\}$, where $d_{ij} = c_i \cdot c_j$, and of the positivity of the factor $a^2 + a^{-2}$.

That $D$ is positive definite is, in turn, elementary:

$$d_{ij} \cdot \frac{c_i}{c_j} = c_i \cdot c_j \cdot \frac{c_i}{c_j} \geq 0.$$ 

Proof of Theorem 2: The sufficiency is immediate for we can write

$$I_A(\alpha_1, \ldots, \alpha_m) = \int_{\mathcal{A}} \prod_{i} F_{x,i}(\alpha_1, \ldots, \alpha_m) \, d\mu(x) \, d\mu(y).$$

To see the necessity, let us assume that $\exists \alpha_i \leq \alpha_i, \ldots, \alpha_m$ so that

$$F_{x,y}(\alpha_1, \ldots, \alpha_m) > F_{x,y}(\bar{\alpha}_1, \ldots, \bar{\alpha}_m)$$

for $(x, y)$ in a set $B \in \Omega \times \Omega$ of positive $d\mu(x) \times d\mu(y)$ measure. Since $F_{x,y} = F_{y,x}$ we see that

$$(x, y) \in B \iff (y, x) \in B,$$

so that $B$ must be of the form $A \times A$ for $A \in \mathcal{A}$ with $\mu(A) > 0$. In particular,

$$I_A(\alpha_1, \ldots, \alpha_m) > I_A(\bar{\alpha}_1, \ldots, \bar{\alpha}_m).$$

Another generalization of Hardy, Littlewood, and Polya’s result is given by the following theorem.

Theorem 4. Let $(\Omega, \mathcal{A}, \mu)$ be a positive measure space with $\mu$ a probability measure. Let $f_i: \Omega \rightarrow \mathbb{R}^+$ be measurable functions. Let

$$(\alpha_1, \ldots, \alpha_m) \in \{(-\infty, 0]^m \cup \{(0, \infty)^m\},$$

then

$$J(\alpha_1, \ldots, \alpha_m) = \int_{\Omega} f_{x,1}(x) \cdots f_{x,m}(x) \, d\mu(x)$$

is log convex as a function of $\alpha = (\alpha_1, \ldots, \alpha_m)$. Moreover, if, in addition, we assume that Condition (y) holds, then all $m^2$ second partials of $\log J$ are nonnegative.

Proof. Let

$$b(x_1, \ldots, x_m) = \prod_{i=1}^{m} f_i^n(x).$$

Consider $K(\alpha_1, \ldots, \alpha_m) = \log J(\alpha_1, \ldots, \alpha_m)$. We have

$$\frac{\partial K}{\partial \alpha_i} = \int_{\Omega} b \log f_i \, d\mu \left\{ \int_{\Omega} b \, d\mu \right\}^{-1}$$

and

$$\frac{\partial^2 K}{\partial \alpha_i \partial \alpha_j} = \left[ \int_{\Omega} b(\log f_i) \log f_j \, d\mu \right] \left[ \int_{\Omega} b \, d\mu \right]^{-2}.$$ 

Interchanging the role of $x$ and $y$ and adding, we find that the numerator on the right-hand side may be expressed as

$$\frac{1}{2} \int_{\Omega \times \Omega} \left[ \log f_i(x) - \log f_i(y) \right] \cdot \left[ \log f_j(x) - \log f_j(y) \right] \, d\nu(x) \, d\nu(y),$$

where we have set

$$d\nu = b \, d\mu \quad \text{(recall $b > 0$)}.$$ 

Condition (y) makes the integrand clearly nonnegative, thus $\partial^2 K / \partial \alpha_i \partial \alpha_j \geq 0$. Setting

$$\log f_i(x) - \log f_i(y) = c_i,$$

we see, as in the proof of Theorem 3, that the positive semidefiniteness of the matrix $\partial^2 K / \partial \alpha_i \partial \alpha_j$ follows immediately from that of the matrix $D = \{d_{ij}\} = \{c_i c_j\}$.