Iterated Differentiable Maps with Nowhere Differentiable Basin Boundaries

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The fractal basin boundary of a two-dimensional discrete dynamical system modelling a chaotic forcing applied to bistability is shown to be identical to the graph of an infinite series \( F(x, t) = \sum_{k=0}^{\infty} c^k \cos(b^k \pi x) \), where \( b \) is an odd integer and \( 0 < a < 1, ab > 1 + \frac{3}{2} \pi \), has finite or infinite differential quotient at any point \( z \in \mathbb{R} \). It is well known that this result has been improved in 1916 by Hardy [3] who has shown that (1) is nowhere differentiable in \( \mathbb{R} \) if \( 0 < a < 1, b > 1 \) and \( ab \geq 1 \). Today, using methods of functional analysis, we even know that the set of functions in \( C_\infty(\mathbb{R}) \) (denoting the set of all bounded continuous real-valued functions on \( \mathbb{R} \) provided with the uniform metric) that are differentiable at at least one point of \( \mathbb{R} \) is of second category in \( C_\infty(\mathbb{R}) \). Thus, functions that are nowhere differentiable are considerably abundant in \( C_\infty(\mathbb{R}) \). Indeed, they are dense in \( C_\infty(\mathbb{R}) \) at least. This result is a consequence of Baire’s category theorem (cf. Brown and Page [4], p. 305).

Fractal structures arise in basin boundaries of analytical maps in the complex plane (i.e. Julia sets and the boundaries of Mandelbrot sets), but also in the nonanalytical case (cf. [5] and refs. therein). Since 1990, some papers even conjecture nowhere-differentiability for boundaries of simple generic two-dimensional maps (cf. [6, 7, 8] and others). However, their assertions solely lean on experimental results.

2. A Conjecture on Nowhere-Differentiability

Recently, Peinke et al. [6] proposed a class of two-dimensional discrete dynamical model systems initiated by the idea that fractal basin boundaries of generic dynamical systems are due to a chaotic forcing applied to bistability (cf. [7, 8]). The map involved reads in general

\[
x_{n+1} = f(x_n),
\]

\[
y_{n+1} = g(y_n) + b x_n \quad (0 < b < 1),
\]

where \( f: [0, 1] \to [0, 1] \) is assumed to be \( C^1 \), unimodal and chaotic, and further \( g: \mathbb{R}^+ \to \mathbb{R}^+ \) is supposed to be \( C^1(\mathbb{R}) \), strictly increasing, bistable with two sinks, 0 and \( \infty \), and one repelling fixed point at 1. Simple examples are given by \( f: x \mapsto 4x(1-x) \) and \( g: y \mapsto y^3 \), respectively.

Obviously, the basin of divergence is given by

\[
D = \{(x_0, y_0) \in [0, 1] \times \mathbb{R}^+_0 \mid x_0^2 + y_0^2 \to \infty \}.
\]

and an easy criterion for divergence is

\[
(x_0, y_0) \in D \iff \exists n \in \mathbb{N}: y_n > 1.
\]

We are interested in the boundary between convergence and divergence in the square \([0, 1] \times [0, 1]\) of the first quadrant.

The boundary of \( D \) is given by

\[
\{(x, Z(x)) \in [0, 1] \times \mathbb{R}^+_0 \mid Z(x) = \lim_{n \to \infty} Z_n(x) \},
\]

for boundaries of simple generic two-dimensional maps (cf. [6, 7, 8] and others). However, their assertions solely lean on experimental results.
Fig. 1. Boundary function $Z$ (complement of $D$ colored black) from (4) for $f: x \mapsto rx(1-x)$ and $g: y \mapsto y^a$ with

a) $b = 0.1, \ r = 4, \ \lambda' - \lambda^a > 0$;  

b) $b = 0.1, \ r = 3.6, \ \lambda' - \lambda^a < 0$.

Fig. 2. Weierstrass function $F$ from (9) for $f: x \mapsto 4x(1-x)$ and $t = 0.9$.

Fig. 3. Weierstrass function $F$ from (9) for $f: x \mapsto 3.6x(1-x)$ and $t = 0.9$. 
where \( Z_n(x) \) can be defined by recursion, i.e.
\[
\begin{align*}
Z_1(x) &= g^{-1}(1 - bx), \\
Z_{n+1}(x) &= g^{-1}(Z_n(f(x)) - bx),
\end{align*}
\]
(5)
\( x \in [0, 1] \), \( n \in \mathbb{N} \). Explicitly, \( Z_n \) is given by the formula (cf. \[6\])
\[
Z_n(x) = g^{-1}(g^{-1}(\ldots g^{-1}(1 - b f^{(n-1)}(x))
\]
\( n \) times
\[
- b f^{(n-2)}(x) \ldots - b f(x) - bx).
\]
(6)
Model assumptions yield for each \( n \in \mathbb{N} \) and \( x \in [0, 1] \):

(a) \( Z_n(0) = 1 \), \( Z_n(1) = g^{-1}(1 - b) \in (0, 1) \),

(b) \( Z_{n+1}(x) \leq Z_n(x) \leq 1 \).

The conjecture in question on \( Z \) reads, roughly spoken (cf. \[6, 8, 9\]):

\( Z \) is nowhere differentiable on \([0, 1]\). (C1)

Numerous numerical simulations have shown experimental evidence to this conjecture (see \[6, 8, 9\] and Figure 1 a). But up to now, any proof is missing.

Defining time means
\[
\lambda^f_n(x) = \frac{1}{n} \sum_{k=0}^{n-1} \ln |f'(f^o(k-1)(x))| \quad (n \in \mathbb{N}),
\]
\[
\lambda^f(x) = \lim_{n \to \infty} \lambda^f_n(x)
\]
(7)
and \( \lambda^g(x) \), \( \lambda(x) \) analogously (with \( g \) instead of \( f \)), conjecture (C1) can be reformulated more precise:

\[ \lambda^f - \lambda^g > 0 \Rightarrow Z \text{ is nowhere differentiable} \]
(and graph \( Z \) is a fractal curve).

The authors of \[6\] additionally assert that \( Z \) is differentiable for \( \lambda^f - \lambda^g < 0 \). They give no proof, but this assertion also seems to be evident from experiments (see Fig. 1 b and refs. from above). Since \( f \) is assumed to be ergodic, \( \lambda^f \) exactly is its Lyapunov (characteristic) exponent and therefore almost everywhere independent of \( x \). \( \lambda^g \) can be interpreted as rate of exponential escape from the boundary graph \( Z \), by no means as a characteristic exponent. Nevertheless, experiments indicate that \( \lambda^g \) should be invariant on \([0, 1]\) as well as \( \lambda^f \).

In an earlier context \[10\], different from the present problem, conjecture (C1) has already been formulated by Okniński. Defining
\[
E_n(x, t) = \sum_{k=0}^{n} t^k f^{o_k}(x) \quad (0 < t < 1),
\]
(8)
where \( f : I \to I \) (\( I \subset \mathbb{R} \) interval) is unimodal and possesses sensitive dependence on initial conditions (i.e. \( \lambda^f > 0 \) on \( I \)), and
\[
F(x, t) = \lim_{n \to \infty} E_n(x, t) = \sum_{k=0}^{\infty} t^k f^{o_k}(x),
\]
(9)
we conjecture \[11\], that for each \( t > \exp(-\lambda^f) \) the transform \( F \) is nowhere differentiable (with respect to \( x \)) at least on an open subset \( X \subset I \) with positive Lebesgue measure (C2). Okniński \[10\] conjectured nowhere-differentiability on \( X = I = (0, 1) \) (but gave an incorrect proof).

A connection between the present boundary map \( Z \) and Okniński’s transform \( F \) can be established with the help of the derivative of \( Z_n \). Some lengthy but straightforward transformations give
\[
Z_n(x) = \sum_{k=1}^{\infty} \sigma_k(x) t^k \exp(k \Delta_k(x)),
\]
(10)
where \( 0 < t < 1 \) and
\[
\sigma_k(x) = \text{sgn} \left( \frac{d}{dx} f^{o_k}(x) \right),
\]
\[
\Delta_k(x) = \lambda^f_k(x) - \lambda^g_k(x).
\]
On the other hand, we get from (7)
\[
E_n(x, t) = \sum_{k=0}^{n} t^k \frac{d}{dx} f^{o_k}(x)
\]
\[
= \sum_{k=0}^{n} \sigma_k(x) t^k \exp(k \lambda^f_k(x)).
\]
(11)
Obviously, (10) and (11) are identical up to the (finite) summation range. Thus, respecting the condition on \( t \), the conjectures (C1) and (C2) are equivalent.
For a special case, where \( f \) is the logistic map
\[
f: x \mapsto rx(1-x)
\]
(12)
in “full chaos” on its domain \([0, 1]\), i.e. \( r = 4 \), conjecture (C2) can be proved, cf. [12]. In this case, for \( t > \exp(-\lambda^2) = \frac{1}{2} \), \( F \) is equal to the Weierstrass function up to a constant (cf. (1) and Fig. 2)
\[
w: z \mapsto \sum_{k=0}^{\infty} t^k \cos(2^k \pi z),
\]
(13)
where \( \pi z = \arccos(1 - 2x) \). But, of course, for values of \( r < 4 \) in the chaotic range [13], graph \( F \) is a fractal curve, too (Fig. 3), and it is (piecewise) smooth else (Fig. 4).

3. About the Converse Problem

The converse question, i.e.
\[
f \text{is nowhere differentiable} \Rightarrow d_H(\text{graph } f) > 1,
\]
where \( d_H \) denotes the Hausdorff-dimension of graph \( f \), has been posed for the first time in 1977 by Mandelbrot [14] for the so-called Mandelbrot-Weierstrass function
\[
m: t \mapsto \sum_{k=-\infty}^{\infty} \frac{1 - \cos(b^k t)}{b^{(2-\delta)k}} \quad (1 < \delta < 2, b > 1). \quad (14)
\]
Tél [15] delivers a scaling relation
\[
m(b t) = b^{2-\delta} m(t)
\]
(15)
(which follows from (14) by a formal replacement of \( n \) by \( n+1 \) according to which the graph of \( m \) on the interval \([t_0, b t_0]\), to arbitrary, can be obtained by magnifying graph \( m \) in the range \([t_0/b, t_0] \) with factors \( b \) in horizontal and \( b^{2-\delta} \) in vertical direction, respectively. This nontrivial symmetry is called self-affinity of graph \( m \) (cf. [15]).

Finally, already in 1980 Berry and Lewis [16] have shown analytically that
\[
d_H(\text{graph } m) = \delta > 1
\]
in the parameter range of (14).

4. Outlook

Any (correct) proof of any conjecture on nowhere-differentiability of a fractal basin boundary gives support to an embedding of fractal structures into analytical mathematics without using tools or methods dealing with their fractal (Hausdorff) dimensions. Therefore, this paper together with [11] and [12] demonstrate a way how to be successful in that direction. Moreover, this method even seems to be generalizable.

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