Onset of Thermosolutal Convection in a Liquid Layer Having Deformable Free Surface – II. Overstability

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The onset of thermosolutal convection driven by a temperature and concentration dependent surface tension is investigated for a thin layer of fluid having a deformable free surface. It is shown that there exist two Crispation numbers (Cr) for oscillatory modes of instability. It is further shown that Cr and the frequency of the oscillatory mode are strongly coupled for large values of Cr. It is found that Cr destabilizes the system for both cases, salted from below and salted from above.

1. Introduction

The effect of a free surface deformation on the onset of a surface tension driven instability in a horizontal thin liquid layer subject to a vertical temperature and concentration gradient is examined using linear stability theory. In [1] we had studied the onset of thermosolutal convection in the frame of stationary convection and obtained the existence of two Crispation numbers Cr1 and Cr2. The other possibility of instability, setting in as overstability, was not explored in that study.

The aim of the present paper is to study the effect of the free surface deformation on the initiation of oscillatory instability.

2. The Problem and its Solution with a Numerical Procedure

The governing equations of W, θ, and φ for linear stability are given by (cf. [1], (13a–c))

\[ (D^2 - a^2) (D^2 - a^2 - \omega P_{-1}^2) W = 0, \]

\[ (D^2 - a^2 - \omega) \theta = -W, \]

\[ \{ \tau (D^2 - a^2) - \omega \} \phi = W. \]  

(1a–c)

The associated boundary conditions are given by (cf. [1], (14a–d) and (15a–e))

\[ W = 0 \quad \text{at} \quad z = 0 \quad \text{(2a–d)} \]

\[ W = DW \quad \text{at} \quad z = 0 \quad \text{(3a–e)} \]

\[ Cr (D^2 - 3a^2 - \omega P_{-1}^2) DW - a^2 (B_0 + a^2) Z = 0, \]

\[ (D^2 + a^2) W + a^2 (M \theta - M' \phi) - a^2 (M + M') Z = 0, \]

\[ (D + B') \theta - B Z = 0, \]

\[ (D + B') \phi + B Z = 0, \]

at \[ z = 1. \]

The effect of a free surface deformation on the onset of oscillatory instability, we shall solve (1)–(3) for \( \omega \neq 0 \). For the boundary conditions (2a, b) at \( z = 0 \) and (3a, b) at \( z = 1 \) we easily find from (1a)

\[ W = A_1 [S_{az} - K_0 C_{az} - (a/b) S_{bz} + K_0 C_{bz}], \]

where

\[ S_i = \sinh i, \quad C_i = \cosh i, \quad (i = a, b, d, e, \text{etc.}), \]

\[ b = [a^2 + \omega / P_i]^{1/2}, \]

\[ K_0 = \frac{(\omega Cr) [b_1 C_a - 2a^2 C_a] + a(B_0 + a^2)[S_a - (a/b) S_b]}{(\omega Cr) [b_1 S_a - 2a b S_b] + a(B_0 + a^2)[C_a - C_b]}, \]

\[ b_1 = 2a^2 + (\omega P_i) \]

and \( A_1 \) is an arbitrary constant.

Similarly we can obtain the solution for \( \theta \) and \( \phi \) by solving the sets (1b, 2c, 3d) and (1c, 2d, 3e) as

\[ \theta = (A_1/\omega) [K_1 S_{az} + (K_0/(1 - P_i)) C_{az} + S_{az} - K_0 C_{az} + (a/b) (P_i/(1 - P_i)) S_{bz}], \]

\[ \phi = (A_1/\omega) [(K_2/\tau) S_{az} - (K_0 \tau/(\tau - P_i)) C_{az} - [S_{az} - K_0 C_{az} + (a P_i/(\tau - P_i) b) S_{bz} - (K_0 P_i/(\tau - P_i)) C_{bz}]. \]

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where \( d = (a^2 + \omega^2)^{1/2} \), \( e = (a^2 + \omega^2)^{1/2} \),

\[
K_1 = \left[ \frac{d_0}{(P_r - 1)} \right] S_d - a C_a + K_0 a S_a
- (a \beta_a (1 - P_r)) S_b
+ B_1 [(K_0)/(P_r - 1)) C_d - (a/b (1 - P_r)) S_b
+ (K_0/(1 - P_r) C_b)]/[d C_d + B_1 S_d],
\]

\[
K_2 = \left[ \frac{d_0 e \tau^2}{(\tau - P_r)} \right] S_e + \tau \left[ a C_a - K_0 a S_a
+ (a \beta_a (1 - P_r)) S_b
+ B_1 [(K_0)/(P_r - 1)) C_d - (a/b (1 - P_r)) S_b
+ (K_0/(1 - P_r) C_b)]/[d C_d + B_1 S_d],
\]

Since \( M \) is real, we have

\[
M' = \frac{a_1 a_5 + a_2 a_6 - a_3 a_1 - a_4 a_2}{a_1^2 + a_2^2}. \quad (12)
\]

Assigning arbitrary values to the parameters \( M' \), \( B_0 \), \( B_1 \), \( B_1' \), \( \tau \), \( P_r \), \( \omega_r \), and \( a \) one can calculate \( M \) from (12) satisfying (11), and the solution is acceptable when (11) is satisfied to within a tolerance error of \( 10^{-5} \). To find the minimum value of \( M = M_c \) we have fixed all the parameters except \( \omega_r \) and \( a \). At first, \( a \) is allowed to vary and we note the minimum value of \( M \), say \( M = M_{1c} \), for a particular value of \( \omega_r \). Then we supply the next value to \( \omega_r \) and vary \( a \) to obtain \( M_{2c} \) and compare \( M_{1c} \) with \( M_{2c} \). In this way finally we obtain \( M_c \), which is the smallest among all \( M_{lc} \) \((i = 1, 2, \ldots) \) for critical values of \( \omega_r (= \omega_{c}) \) and \( a (= a_r) \). It should be pointed out here that for large values of the wave number \( a \) our numerical tolerance is not fulfilled, so we have discarded those results and conclude that the neutral state is stationary for those values of \( a \).

3. Results and Discussion

Figure 1 shows the variation of \( M \) with \( a \) for several values of \( \text{Cr} \) when the other parameters are fixed. It should be pointed out here that \( M \) will be unchanged for all values of \( \text{Cr} \leq 0.00002 \). We designate this \( \text{Cr} \) as \( \text{Cr}_1 \) (see [1]). It is clear from Fig. 1 that, as \( \text{Cr} \) increases past \( \text{Cr}_1 \), a local minimum is formed for small values of \( a \) but \( M \) will be smallest for large values of \( a \). For \( \text{Cr} = \text{Cr}_2 \), a minimum value for \( M \) can be obtained at two different values of \( a \). On further increase of \( \text{Cr} \) past \( \text{Cr}_2 \), \( M \) will be smallest for small values of \( a \). It is evident from Fig. 1 that no oscillatory instability will exist for \( a \rightarrow 0 \). This results confirms our findings in [3]. Figures 2 and 3 depict the change of \( M_{1c} \) with \( \omega_r \). It is clear that the positive or negative value of \( M_{1c} \) depends on the selection of the frequency range of \( \omega_r \), although for the entire range of \( \omega_r \), the smallest value of \( M_{1c} \) (i.e. \( M_c \)) will always be negative. One may interpret this negative value of \( M_c \) as heating from above [4]. Now we like to emphasize that by selecting
Fig. 1. Variation of $M$ with $a$ for different values of $\text{Cr}$ when $P_e = 0.001$, $\tau = 0.07$, $B_0 = 0.1$, $B_i = B'_i = 0$, $M' = 4$ and $\omega = 0.001$.

Fig. 2. Variation of $M_{ie}$ with log ($\omega$) for different $M'$ when $P_e = 0.001$, $\tau = 0.01$, $B_0 = 0.1$, $B_i = B'_i = 0$, $\text{Cr} = 0.001$. — for $B_0 = 0.1$, $M' = 20$ and - - - for $M' = -5$.

Fig. 3. Variation of $M_{ie}$ with log ($\omega$) for $P_e = 0.001$, $\tau = 0.01$, $B_0 = 0.1$, $B_i = B'_i = 0$, $\text{Cr} = 0.001$ when $M' = 20$.

Fig. 4. Variation of $\omega$ with $\text{Cr}$ when $\tau = 0.01$, $P_e = 0.001$, $B_0 = 0.1$, $M' = 20$ and $B_i = B'_i = 0$. 

$\omega \times 10^4$

$M_{ie} \times 10^4$

$\text{cr} \times 10^3$

$\log(\omega)$
a particular frequency range for $\omega_1$ it is possible to suppress the onset of the oscillatory mode of instability either for the system heated below or from above. Figure 2 shows the changes of $M_c$ with $\omega_1$ for salted from below, whereas Fig. 3 shows the same for salted from above. Figure 4 represents the variation of $\omega_*$ (the value of $\omega_1$ for which $|M_c|$ is minimum) with $Cr$ for a particular set of values of all other parameters. It is clear from the graph that $\omega$ increases with $Cr$ up to a certain value of $Cr$ (say $Cr^*$), beyond which $\omega_*$ oscillates within a certain range. The physical explanation for the oscillation of $\omega_*$ for $Cr > Cr^*$ is not clear to us. However, Fig. 5 depicts the variation of $M_c$ with $Cr$ for fixed values of $M'$, $P_r$, $B_0$, $\tau$, $B_i$, and $B_l$. Here we have taken that value of $\omega_*$ for which $Cr = Cr^*$ as predicted in Figure 4. Figure 6 shows the change of $-M_c$ with $M'$ for different $Cr$ keeping all other parameters fixed. It is clear that the increase of $Cr$ destabilizes the system for both cases, salted from below or above.