Localized Ideal and Resistive Instabilities in 3-dimensional MHD Equilibria with Closed Field Lines

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A class of stability criteria is derived for MHD equilibria with closed field lines. The destabilizing perturbations have finite gradients along the field and are localized around a field line, the localization being stronger on the pressure surface than in the radial direction. By contrast, in sheared configurations the localization is comparable in both directions. The derived stability criteria are less stringent than those obtained for MHD equilibria with shear for similarly localized perturbations in the limit of low shear. These results, obtained from the energy principle, are a particular case of those obtained by solving the linearized resistive MHD equations with an appropriate ansatz and subsequently taking the limit of vanishing shear and resistivity.

In previous papers [1–3], criteria for the stability of symmetric and also of nonsymmetric MHD equilibria with shear in respect of ideal and resistive localized perturbations were derived. For the sake of completeness, and owing to the interest given to shearless configurations, a class of stability criteria for localized perturbations is presented here. The derivation clearly shows the differences (and also the similarities) of the destabilizing perturbations in the cases with and without shear.

The perturbations considered here leave the plasma surface unchanged. For these, the change $\delta W$ in potential energy is

$$\delta W = \frac{1}{2} \left[ \frac{\mathbf{Q} \times \mathbf{B}}{B^2} + \frac{|\mathbf{Q} \cdot \mathbf{B} - \mathbf{\xi} \cdot \mathbf{Vp}|^2}{B^2} + \frac{\gamma_{\mu}}{\mu} |\mathbf{V} \cdot \mathbf{\xi}|^2 \right] + \frac{B \cdot J}{2B^2} \left[ (\mathbf{B} \times \mathbf{\xi}) \cdot \mathbf{Q} + (\mathbf{\xi} \times \mathbf{B}) \cdot \mathbf{Q}^* \right]
$$

$$- (\mathbf{\xi} \cdot \mathbf{Vp}) (\mathbf{\xi} \cdot \mathbf{x}) + (\mathbf{\xi}^* \cdot \mathbf{Vp}) (\mathbf{\xi} \cdot \mathbf{x}) \right] \text{d} \tau,$$

where all the symbols have their usual meaning and are explained in detail elsewhere [1–3].

To describe the system, coordinates $u, \theta, \zeta$, $\varphi = \zeta - \zeta \vartheta$ are employed, where $u, \theta, \zeta$ are Hamada coordinates and $q = \Psi / J = M / N$ ($M, N$ integers) is the safety factor, which is the same for all field lines. In these coordinates, the physical quantities are periodic in $\theta$ with period $N$, and in $\varphi$ with period 1. At constant $\varphi$ and $u, \theta$ measures the angle along a particular field line, which closes upon itself after a poloidal angle of $\Delta \theta = N$.

The magnetic field $\mathbf{B}$ and the gradient along a field line are given by $\mathbf{B} = \mathbf{\nabla} \varphi \times \mathbf{V} \varphi$ and $\mathbf{B} \cdot \mathbf{\nabla} = \mathbf{\nabla} \varphi \times \mathbf{V} \varphi$, respectively. Setting

$$\xi = U \mathbf{\nabla} \varphi \times \mathbf{V} \varphi + T \mathbf{V} \varphi \times \mathbf{V} \varphi + S \mathbf{\nabla} \varphi \times \mathbf{V} \varphi,$$

one obtains

$$\mathbf{Q} \times \mathbf{B} = \mathbf{\xi} \left[ \frac{\partial U}{\partial \varphi} - \frac{\partial T}{\partial \varphi} - \frac{\partial S}{\partial \varphi} \right],$$

$$\mathbf{Q} \cdot \mathbf{B} - \mathbf{\xi} \cdot \mathbf{Vp} = -B^2 \left[ 2 (U x_\varphi + T x_\varphi) + \frac{\partial U}{\partial \varphi} \right]
$$

$$+ \frac{\partial T}{\partial \varphi} + \frac{\partial S}{\partial \varphi} + \frac{1}{2B^2} \left[ (B \times \mathbf{\xi}) \cdot \mathbf{Q} + (\mathbf{\xi} \times \mathbf{B}) \cdot \mathbf{Q}^* \right],$$

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where all the symbols have their usual meaning and are explained in detail elsewhere [1–3].

The perturbations $\xi$ considered here are localized around a field line defined by $u = u_0, \varphi = \varphi_0$. Setting

$$t = (u - u_0)/v_0, \quad x = (\varphi - \varphi_0)/v_2,$$

with $v_2 \ll 1$, we require that $t \sim x \sim O(1)$, when applied to perturbations. Furthermore, the perturbations are required to vanish for $|u - u_0| \geq v_0, \varphi - \varphi_0 \geq v_2$, i.e.
Notice that, contrary to the case with shear, the perturbations are less localized in the radial direction.

The equilibrium quantities $A(v, \theta, \varphi)$ are now expanded in a Taylor series around $v = v_0$, $\varphi = \varphi_0$:

$$A(v, \theta, \varphi) = A(v_0, \theta_0, \varphi_0) + \frac{\partial A}{\partial v}(\theta_0, \varphi_0)(v - v_0) + O(\varepsilon^2),$$

and the test functions $U, T, S$ in series of the form

$$U = U_0 + U_1 \varepsilon + U_2 \varepsilon^2 + \ldots,$$

where the order of magnitude of the different terms is given by the powers of $\varepsilon$, and the functions $U_i$ may depend implicitly on $\varepsilon$ (similarly for $T$ and $S$). Then, from the lowest order $\delta W$, the following conditions are obtained: $T_0 = 0$ and $\partial_\theta U_0 + v_0 \partial_\theta T_0 = 0$ (otherwise, $\delta W$ will be positive).

Then, in this approximation, one has

$$Q \times B = \dot{\chi}^2 \frac{\partial U_0}{\partial \theta} \phi + O(\varepsilon),$$

$$Q \cdot B - \chi \cdot \nabla p = -B^2 \left[ 2 U_0 \chi_0 + \frac{1}{v_0} \frac{\partial U_1}{\partial t} + \frac{\partial T_2}{\partial \theta} \right] + O(\varepsilon),$$

$$\nabla \cdot \xi = \frac{1}{v_0} \frac{\partial U_1}{\partial t} + \frac{\partial T_2}{\partial \theta} + \dot{\chi} \frac{\partial \phi_0}{\partial \theta} + O(\varepsilon),$$

$$B \times \xi^* \cdot Q = O(\varepsilon),$$

$$\langle \xi \cdot \nabla p \rangle \langle \xi^* \cdot \phi \rangle = \dot{\rho} \chi_0 |U_0|^2 + O(\varepsilon).$$

The function $S_0$ only appears in the term $\nabla \cdot \xi$ and can be used in the minimalization process to eliminate the contribution of the oscillating part (oscillating in $\theta$) of $h := \partial U_1 / v_0 \partial t + \partial T_2 / \partial \theta$. This is different in the case with shear, where the term $\nabla \cdot \xi$ can be completely eliminated. Then, subsequent minimalization of $\delta W$ with respect to $h$ leads, in lowest order in $\varepsilon$, to the following expression for the potential energy:

$$\delta W_0(v_0, \varphi_0) = \frac{v_0^3}{2} \int_{-1}^{1} \int_{-1}^{1} \int_{0}^{N} d\theta \left[ \frac{\dot{\chi}^2}{B^2} |\nabla \phi|^2 \left( \frac{\partial U_0}{\partial \theta} \right)^2 \right]$$

$$+ \frac{4 \gamma_H P}{\langle 1 + \gamma_H P \rangle} |\langle \chi_0 U_0 \rangle|^2 - 2 \dot{\rho} \chi_0 |U_0|^2.$$
the simplest way, we consider the previously derived equations governing resistive localized modes in sheared equilibria. These are Eqs. (13)–(19) of references [2, 3]. In these equations, the shear $\dot{q}$ is now set to 0, and it is assumed that the radial gradients of the perturbations are much smaller than the $x$-derivatives. (Notice that $t$ and $x$ are defined in [2, 3] in a slightly different way). The appropriate ansatz to solve these equations is then

$$\Phi(t, \vartheta, x) = f(e^t, e^2 x) e^{i\alpha x} \mathcal{F}(\vartheta),$$

(17)

$$\delta B \cdot B_0(t, \vartheta, x) = f(e^t, e^2 x) e^{i\alpha x} b(\vartheta),$$

(18)

$$i z \dot{v}_0 D(\vartheta) := b(\vartheta) - i z \dot{v}_0 F(\vartheta),$$

(19)

and the stability problem of resistive localized modes in equilibria with closed field lines can be reduced to solving two coupled differential equations for $F(\vartheta)$ and $D(\vartheta)$, namely

$$\frac{d}{d\vartheta} \left[ \frac{|\nabla \Phi|^2}{B^2} \frac{dF}{|\nabla \Phi|^2} \right] - \frac{\dot{q} \gamma^2}{\dot{\vartheta}^2 B^2} |\nabla \Phi|^2 F = - \frac{2 \dot{\vartheta}}{\dot{\vartheta}^4} \chi_v (F + D),$$

(20)

$$\frac{d}{d\vartheta} \left[ \frac{1}{B^2} \frac{dD}{d\vartheta} \right] - \frac{\dot{q} \gamma^2}{\dot{\vartheta}^2 B} \left( \frac{\gamma_H p + B^2}{\gamma_H p B^2} \right) D$$

$$- \frac{\dot{q} \gamma^2}{\dot{\vartheta}^2 B^2} \eta^* \gamma |\nabla \Phi|^2 D - \frac{2 \dot{\vartheta}}{\dot{\vartheta}^4} \chi_v (D + F)$$

$$= \frac{2 \dot{\vartheta}}{\dot{\vartheta}^2} \frac{\gamma^2}{\gamma_H p} \chi_v F.$$  

(21)

These equations are formally the same as those describing resistive ballooning modes in equilibria with shear, with $\dot{q}$ set to 0. The crucial difference is that here the equations must be solved with different boundary conditions, namely $F(\vartheta + N) = F(\vartheta)$, $D(\vartheta + N) = D(\vartheta)$, and not in the infinite interval.

Setting $\eta^* = 0$ yields

$$\frac{d}{d\vartheta} \left[ \frac{|\nabla \Phi|^2}{B^2} \frac{dF}{|\nabla \Phi|^2} \right] - \frac{\dot{q} \gamma^2}{\dot{\vartheta}^2 B^2} |\nabla \Phi|^2 F = - \frac{2 \dot{\vartheta}}{\dot{\vartheta}^4} \chi_v (F + D),$$

(22)

$$\frac{d}{d\vartheta} \left[ \frac{1}{B^2} \frac{dD}{d\vartheta} \right] - \frac{\dot{q} \gamma^2}{\dot{\vartheta}^2 B} \left( \frac{\gamma_H p + B^2}{\gamma_H p B^2} \right) D = \frac{2 \dot{\vartheta}}{\dot{\vartheta}^2} \frac{\gamma^2}{\gamma_H p} \chi_v F,$$

(23)

which are the appropriate equations to describe ideal ballooning modes in equilibria with closed field lines.

Near marginal stability, the growth rate $\gamma$ is small ($\gamma^{-1}$ is large compared with the time needed for sound and Alfvén wave propagation). In this limit, and introducing $\tilde{D}$ by

$$\tilde{D} := D - \frac{2 \dot{\vartheta}}{\dot{\vartheta}^2} \frac{\gamma_H p}{\gamma} \left( \frac{1}{B^2} \right) \left( \frac{\gamma_H p + B^2}{\gamma_H p B^2} \right) \langle \chi_v F \rangle,$$

(24)

one can obtain from (22) and (23) the relation

$$\gamma^2 \int_0^N |r|^2 d\vartheta = \int_0^N \left[ - \frac{|\nabla \Phi|^2}{B^2} \frac{dF}{|\nabla \Phi|^2} \right]^2 + \frac{2 \dot{\vartheta}}{\dot{\vartheta}^4} \chi_v F^2$$

$$- \frac{4 \dot{\vartheta}}{\dot{\vartheta}^4} \left( \frac{\gamma_H p}{\gamma_H p B^2} \right) \langle \chi_v F \rangle \chi_v F \frac{d\vartheta}{d\vartheta},$$

(25)

where $|r|^2$ is a positive definite function of $F$ and $d\tilde{D}/d\vartheta$. It is clear that the sign of $\gamma^2$ and that of $-A$ are equal, and (25) is equivalent to (15).