Hydrogen in Electrodynamics.
V. Dirac-Hydrogen by Means of New Operators

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After a discussion of the one-component Schrödinger (1926) and the four-component Dirac (1928) representation of hydrogen it is shown that the six-component electrodynamic picture is considerably simpler and clearer. The computational effort is reduced to a fraction.

In preparation for the general electrodynamic the Dirac side of (1) reads explicitly

\[
\frac{2\pi i}{hc} \left( m_0 c^2 + hv + \frac{e^2_0}{r} \right) \psi_1 + (\hat{c}_1 - i \hat{c}_2) \psi_4 + \hat{c}_3 \psi_3 = 0,
\]

(5)

\[
\frac{2\pi i}{hc} \left( m_0 c^2 + hv + \frac{e^2_0}{r} \right) \psi_2 + (\hat{c}_1 + i \hat{c}_2) \psi_3 - \hat{c}_3 \psi_4 = 0,
\]

(6)

\[
\frac{2\pi i}{hc} \left( -m_0 c^2 + hv + \frac{e^2_0}{r} \right) \psi_3 + (\hat{c}_1 - i \hat{c}_2) \psi_2 + \hat{c}_3 \psi_1 = 0,
\]

(7)

\[
\frac{2\pi i}{hc} \left( -m_0 c^2 + hv + \frac{e^2_0}{r} \right) \psi_4 + (\hat{c}_1 + i \hat{c}_2) \psi_1 - \hat{c}_3 \psi_2 = 0.
\]

(8)

If, for clarity, we put

\[
i \omega/c = \frac{2\pi i}{hc} \left( m_0 c^2 + hv + \frac{e^2_0}{r} \right)
\]

(9)

and

\[
i \omega/c = \frac{2\pi i}{hc} \left( -m_0 c^2 + hv + \frac{e^2_0}{r} \right)
\]

(10)

we now get, instead of (5)–(8), the system

\[
i \omega/c \psi_1 + (\hat{c}_1 - i \hat{c}_2) \psi_4 + \hat{c}_3 \psi_3 = 0,
\]

(11)

\[
i \omega/c \psi_2 + (\hat{c}_1 + i \hat{c}_2) \psi_3 - \hat{c}_3 \psi_4 = 0,
\]

(12)

\[
i \omega/c \mu \psi_3 + (\hat{c}_1 - i \hat{c}_2) \psi_2 + \hat{c}_3 \psi_1 = 0,
\]

(13)

\[
i \omega/c \mu \psi_4 + (\hat{c}_1 + i \hat{c}_2) \psi_1 - \hat{c}_3 \psi_2 = 0.
\]

(14)

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For the general d’Alembert solution [1] (26) we need the two partial solutions, separated in radial and angular variables,

\[ \psi_k = \psi_k e^{-i\omega t} = C_k R_k P_k^{m_k} e^{\pm im_k \phi} e^{-i\omega t} \] (15)

or, respectively, the two amplitudes

\[ \psi_k = C_k R_k P_k^{m_k} e^{\pm im_k \phi} \] (16)

We now solve the system (11)—(14) by inserting the ansatz (16) with the lower index, that is

\[ \tilde{\psi}_k = \tilde{C}_k R_k P_k^{m_k} e^{\pm im_k \phi} \] (17)

into (11) and (13). Using (2) and (3) we get

\[ i \varepsilon \omega \tilde{C}_1 R_1 P_1^m \tilde{e}^{-e^{-i(m+1)}} + \tilde{C}_4 e^{-i(m+1)} \]

\[ = \frac{2l_4 + 1 - R_{4, l_4 + 1} P_{l_4 - 1}^{m_4 + 1} + R_{l_4 - 1} P_{l_4 + 1}^{m_4 + 1}}{2l_4 + 1} \]

\[ \tilde{C}_3 e^{-i(m+1)} \]

\[ \tilde{C}_2 e^{-i(m+1)} \]

\[ \tilde{C}_1 e^{-i(m+1)} \]

\[ \frac{[R_{1, l_1 + 1} P_1^{m_1 + 1} + R_{l_1 - 1} P_{l_1 + 1}^{m_1 + 1}]}{2l_1 + 1} = 0 \] (18)

\[ i \mu \omega \tilde{C}_3 R_3 P_3^m \tilde{e}^{-e^{-i(m+1)}} + \tilde{C}_4 e^{-i(m+1)} \]

\[ = \frac{2l_2 + 1 - R_{2, l_2 + 1} P_{l_2 - 1}^{m_2 + 1} + R_{l_2 - 1} P_{l_2 + 1}^{m_2 + 1}}{2l_2 + 1} \]

\[ \tilde{C}_3 e^{-i(m+1)} \]

\[ \tilde{C}_2 e^{-i(m+1)} \]

\[ \tilde{C}_1 e^{-i(m+1)} \]

\[ \frac{[R_{1, l_1 + 1} P_1^{m_1 + 1} + R_{l_1 - 1} P_{l_1 + 1}^{m_1 + 1}]}{2l_1 + 1} = 0 \] (19)

Here we remember from the derivation [2] of (2) and (3) that the comma-differentiation in the lower index of the radial functions \( R_k \) is given by the operator equation

\[ R_{k,a} = \left( \frac{d}{r} + \frac{a}{r} \right) R_k \] (20)

according to [2] (17). From (18)—(19), with

\[ m_4 + 1 = m_2 + 1 = m_1 = m_3 = m + 1, \]

\[ l_4 = l_2 = l_H, \]

\[ R_3 = R_4 = R_H, \]

\[ l_1 = l_2 = l_E, \]

\[ R_1 = R_2 = R_E \]

we deduce the system

\[ i \varepsilon \omega \tilde{C}_1 R_1 P_1^{m+1} \]

\[ + \tilde{C}_4 (-R_{H, l_H + 1} P_{l_H - 1}^{m+1} + R_{H, -l_H} P_{l_H + 1}^{m+1}) \]

\[ + \tilde{C}_3 [R_{H, l_H + 1} P_{l_H - 1}^{m+1} (l_H + m + 1) \]

\[ + R_{H, -l_H} P_{l_H + 1}^{m+1} (l_H - m)] = 0, \] (21)

\[ i \mu \omega \tilde{C}_3 R_3 P_3^{m+1} \]

\[ + \tilde{C}_4 (-R_{E, l_E + 1} P_{l_E - 1}^{m+1} + R_{E, -l_E} P_{l_E + 1}^{m+1}) \]

\[ + \tilde{C}_3 [R_{E, l_E + 1} P_{l_E - 1}^{m+1} (l_E + m + 1) \]

\[ + R_{E, -l_E} P_{l_E + 1}^{m+1} (l_E - m)] = 0. \] (22)

The notation \( /_E, /_H, R_E, R_H \) is arbitrary and can be replaced or interchanged at will; it has been chosen in this way only with regard to a subsequent general electrodynamic hydrogen solution. - In the system (24)—(25) two alternatives are visible: We may either put

\[ C_4 = C_3 (l_H + m + 1), \]

\[ C_2 = -C_1 (l_E - m) \] (26)

(2) They possess the hydrogen spectrum.
For the case (27), on the other hand, we get
\[ i\frac{\omega}{c} \tilde{C}^{\Pi}_1 R^{\Pi}_E P^{m+1}_{l_H} + \tilde{C}^{\Pi}_3 R^{\Pi}_H P^{m+1}_{l_H} = 0, \] (33)
\[ -\mu \frac{\omega}{c} \tilde{C}^{\Pi}_3 R^{\Pi}_H P^{m+1}_{l_H} + i \tilde{C}^{\Pi}_1 R^{\Pi}_E P^{m+1}_{l_H} = 0, \] (34)
and by finally fixing
\[ i\tilde{C}^{\Pi}_1 = \tilde{C}^{\Pi}_3 = \tilde{C}^{\Pi}, \quad l^{\Pi}_H = l^{\Pi}_E + 1 = l^{\Pi}, \] (35)
we have arrived at the second pair of Dirac's R-equations
\[ e \frac{\omega}{c} R^{\Pi}_E + R^{\Pi}_H l^{\Pi}_m = 0, \] (36)
\[ -\mu \frac{\omega}{c} R^{\Pi}_H + R^{\Pi}_E l^{\Pi}_m = 0, \] (37)
which also possess the hydrogen spectrum.

Turning now to the ansatz (16) with the upper index, that is
\[ + \psi_k = C_k R_k P^{m_k} e^{i m_k \phi}, \] (38)
we can proceed analogously. We insert (38) into (12) and (14), again using (2) and (3), and transform the pair of equations obtained in that way by means of
\[ m_3 + 1 = m_4 = 3, \quad l_3 = l_4 = l_H, \quad R_3 = R_4 = R_H, \] (39)
\[ l_1 = l_2 = l_E, \quad R_1 = R_2 = R_E \] (40)
into a pair of equations that doesn't contain the azimuthal angle any more. The, instead of (26) and (27), we have the two alternatives
\[ +C_3 = C_4 (l_H + m + 1), \quad +C_1 = C_2 (l_E - m) \] (42)
or
\[ +C_3 = C_4 (l_H - m), \quad +C_1 = -C_2 (l_E + m + 1) \] (43)
Besides, instead of (30), we now have
\[ -i +C_2 = C_4 = C_1, \quad l^{\Pi}_H = l^{\Pi}_E - 1 = l^{\Pi} \] (44)
or
\[ -i C_2 = C_4 = C_1, \quad l^{\Pi}_H = l^{\Pi}_E + 1 = l^{\Pi}, \] (45)
after which the two pairs of equations again go over into the two pairs of R-equations (31)–(32) and (36)–(37).

Looking back and summarizing we arrive at the following constants and solutions:
\[ \tilde{C}^{\Pi} = C^I \begin{pmatrix} -i & 1 \\ 1 & -i \end{pmatrix}, \quad C^{\Pi} = C^I \begin{pmatrix} -i & -i \\ 1 & -i \end{pmatrix} \] (46)
for
\[ \tilde{\varphi}^I = C^I \begin{pmatrix} i(l^I - m + 1) & i(l^I + m) \\ 1 & 1 \end{pmatrix}, \quad \varphi^I = C^I \begin{pmatrix} i(l^I - m + 1) & i(l^I + m) \\ 1 & 1 \end{pmatrix} \] (47)
\[ \tilde{\varphi}^{\Pi} = C^{\Pi} \begin{pmatrix} -i & 1 \\ 1 & -i \end{pmatrix}, \quad \varphi^{\Pi} = C^{\Pi} \begin{pmatrix} -i & -i \\ 1 & -i \end{pmatrix} \] (48)
\[ \tilde{\varphi}^I = C^I \begin{pmatrix} i(l^I - m + 1) & i(l^I + m + 1) \\ 1 & 1 \end{pmatrix}, \quad \varphi^I = C^I \begin{pmatrix} i(l^I - m + 1) & i(l^I + m + 1) \\ 1 & 1 \end{pmatrix} \] (49)
\[ \tilde{\varphi}^{\Pi} = C^{\Pi} \begin{pmatrix} -i & 1 \\ 1 & -i \end{pmatrix}, \quad \varphi^{\Pi} = C^{\Pi} \begin{pmatrix} -i & -i \\ 1 & -i \end{pmatrix} \] (50)