Negative Energy Waves in the Framework of Vlasov-Maxwell Theory

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On the basis of a variational formulation of the Vlasov-Maxwell theory it was recently shown that, for instance, all magnetically confined plasmas allow the existence of negative energy waves. Such waves can become nonlinearly and dissipatively unstable and might therefore be of importance in explaining anomalous transport. The proof of this result uses infinitely strongly localized perturbations. This is, however, not necessary: in this paper it is shown by discussing general, homogeneous, magnetized plasmas that the necessary localization is related to the average gyroradius $r_g$ of the relevant particle species. For unstable plasmas the extent or wavelengths of negative energy waves can be of the order of $r_g$, whereas for linearly stable plasmas the extent can be a small fraction of $r_g$.

1. Introduction

Linear negative energy waves are of interest in the context of nonlinear and dissipative instabilities [1,2]. They might also have a bearing on, for example, problems of anomalous transport. This paper presents conditions for the existence of negative energy waves derived via a variational formulation of the Vlasov-Maxwell theory [3–5].

In [5] Noether’s theorem is used to obtain a second-order energy expression in the perturbations which is the wave energy. Using localized perturbations, it is shown there for any equilibrium that negative energy waves exist if for at least one particle species $v$

$$k \cdot v - \nabla f(0)/\nabla \alpha > 0$$

holds for some $\alpha$ and $x$ and for some directions $k$. This generalizes a result recently obtained by Morrison for homogeneous isotropic plasmas [2]. Here $f_\alpha(0)$ is the unperturbed distribution function for species $v$ in a frame of reference in which the total energy of the equilibrium is smallest.

Since the Vlasov theory becomes invalid for length scales smaller than the Debye length, the question is to what degree it is necessary to localize the perturbations. For a homogeneous plasma with $B(0) = 0$ no localization is necessary. For general inhomogeneous systems with $B(0) \neq 0$ the localization needed should be similar to that for general homogeneous magnetized plasmas, which are therefore investigated in this paper.

2. Second-order Wave Energy for a Homogeneous Magnetized Plasma

The constant unperturbed magnetic field $B(0)$ and the unperturbed vector potential $A(0)$ are taken as

$$B(0) = (0, 0, B(0)), \quad A(0) = (0, B(0) x, 0).$$

In [5] it is shown that for $B(0) = 0$ the minimum of the second-order wave energy $E^{(2)}$ is obtained for a vanishing perturbation $A(0)$ of the vector potential:

$$A^{(0)} = 0$$

which is a possible choice in the sense of an initial condition. In this paper (3) is also chosen for $B(0) \neq 0$, which, however, might no longer correspond to the minimum of $E^{(2)}$ and therefore overestimate the necessary localization. Further we take

$$|F^{(0)}_v|^2 \sim \delta(v_x)$$

which is possible because of $f_\alpha^{(0)} = f_\alpha(0)(v_x^2 + v_y^2, v_z)$. With (2), (3) and (4) the second-order wave energy derived in [5] becomes

$$E^{(2)} = \sum_v \frac{1}{m_v} \int \left[ \frac{1}{2} \partial^2 c \frac{\partial F^{(0)}_v}{\partial \alpha} - \alpha \frac{\partial F^{(0)}_v}{\partial \alpha} e_x \right]^2 + \left[ \frac{1}{2} \alpha \frac{\partial F^{(s)}_v}{\partial \alpha} - \frac{\partial F^{(s)}_v}{\partial \alpha} e_x \right] \cdot \left[ \frac{1}{2} \alpha \frac{\partial F^{(s)}_v}{\partial \alpha} + \text{conj. compl} \right] \right]$$

$$+ \frac{1}{16 \pi} \int \left[ 1 \frac{1}{2} \partial^2 c |F^{(s)}_v|^2, \right.$$

$$\omega_c = e_v B(0)/(m_v c),$$

$e_v$ is the unit vector in the x-direction, $E^{(s)}$ is the complex amplitude of the electric field perturbation and

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$F_v^{(s)}(x,v,t)$ is a generating function for the perturbation of the particle positions and velocities:

$$x_v^{(s)} = \frac{1}{m_v} \frac{\partial F_v^{(s)}}{\partial v} , \quad v_v^{(s)} = -\frac{1}{m_v} \left( \frac{\partial F_v^{(s)}}{\partial x} - \omega_v \frac{\partial F_v^{(s)}}{\partial v} e_v \right). \tag{6}$$

The electric field energy can be made equal to zero, being a bilinear expression of phase space integrals involving linearly the functions $F_v^{(s)}$, without influencing the particle contributions to $E^{(2)}$, see [5]. With

$$F_v^{(s)} \sim e^{i k \cdot x}$$

relation (5) becomes

$$E^{(2)} = V \sum \frac{1}{v} \int d^3 v \left[ -|F_v^{(s)}|^2 \mathbf{k} \cdot \mathbf{v} \cdot \frac{\partial F_v^{(s)}}{\partial v} \right. \left. + i \omega_v k_x f_v^{(0)} \frac{\partial F_v^{(s)}}{\partial v_y} \right. \left. + i \omega_v v_x \left( k \cdot \frac{\partial f_v^{(0)}}{\partial v_y} F_v^{(s)} - \frac{\partial f_v^{(0)} F_v^{(s)}}{\partial v_y} k \cdot \frac{\partial F_v^{(s)}}{\partial v} \right) \right. \left. + \omega_v^2 f_v^{(0)} \frac{\partial F_v^{(s)}}{\partial v_y} \right]. \tag{7}$$

$V$ is a large periodicity volume. $|\mathbf{k}| \to \infty$ again yields the general condition (1). For finite $k$ we discuss separately parallel and perpendicular wave propagation.

### 3. Wave Propagation Parallel to $B^{(0)}$

In this case

$$\frac{\partial F_v^{(s)}}{\partial v_y} = 0 , \quad v_z \frac{\partial}{\partial v_z} \int f_v^{(0)}(v_z^2,v_z) \, dv_z > 0 \tag{8}$$

allows $E^{(2)} < 0$ without any restriction on $|\mathbf{k}|$.

In general, one will have conditions between (1) and (8). This is qualitatively similar to perpendicular wave propagation, which is treated more explicitly in the next section.

### 4. Wave Propagation Perpendicular to $B^{(0)}$

For this case (7) becomes with $\mathbf{k} = (0,k,0)$

$$E^{(2)} = V \sum \frac{1}{v} \frac{1}{4m_v} \int d^3 v \left[ \omega_v^2 f_v^{(0)} \frac{\partial F_v^{(s)}}{\partial v_y} - |F_v^{(s)}|^2 k_v^2 v_y \frac{\partial f_v^{(0)}}{\partial v_y} \right]. \tag{9}$$

From (9) it follows that

$$k \frac{\partial F_v^{(s)}}{\partial v_y} \neq 0 \tag{10}$$

is necessary for $E^{(2)} < 0$. Minimization of $E^{(2)}$ under the constraint

$$V \sum \frac{1}{v} \frac{1}{4m_v} \int d^3 v \omega_v^2 f_v^{(0)} \left| \frac{\partial F_v^{(s)}}{\partial v_y} \right|^2 = 1 \tag{11}$$

with a Lagrange parameter $1/\lambda - 1$ yields for the minimum of $E^{(2)}$:

$$E^{(2)} = 1 - 1/\lambda. \tag{12}$$

This is negative for $0 < \lambda < 1$. With the dimensionless quantities

$$\hat{v}_v = \frac{v_v}{v_{th,v}}, \quad \hat{v} = v_{th,v}/\omega_v = k^2 r_{gv}^2 = \Lambda, \tag{13}$$

the corresponding Hermitian eigenvalue problem is

$$A \hat{v}_y \frac{\partial f_v^{(0)}}{\partial \hat{v}_y} F_v^{(s)} + \frac{\partial}{\partial \hat{v}_y} \left( f_v^{(0)} \frac{\partial F_v^{(s)}}{\partial \hat{v}_y} \right) = 0. \tag{14}$$

Negative $E^{(2)}$ are associated with

$$k^2 r_{gv}^2 = \Lambda/\hat{\Lambda} > \Lambda > 0. \tag{15}$$

The smallest $k$ and therefore the least localization corresponds to the smallest eigenvalue $\Lambda = \Lambda_{\min}$:

$$k_{\min} r_{gv} = \sqrt{\Lambda_{\min}}. \tag{16}$$

$\Lambda > 0$ requires that $f_v^{(0)}$ have a minimum with respect to $\hat{v} = \sqrt{\hat{v}_x^2 + \hat{v}_y^2}$. The minimum wave vector is then qualitatively given by

$$k_{\min} r_{gv}^2 \sim \frac{1}{A \hat{v}_y \hat{v}_v} \left( f_v^{\max} + f_v^{\min} \right), \tag{17}$$

$f_v^{\max}$: the maximum of $f_v^{(0)}(v_{x}^2,v_{y})$, $f_v^{\min}$: its relative minimum in $A \hat{v}_y$, $A \hat{v}_y$: distance between the maximum and the minimum of $f_v^{(0)}$ with respect to $\hat{v}_y$, $\hat{v}_v$: velocity somewhere between the maximum and the minimum of $f_v^{(0)}$.

A numerical solution of the eigenvalue problem (14), (11) was obtained for

$$f_v^{(0)}(v_{x}^2,v_{y}) = \frac{1 - \gamma/\beta}{(2\pi)^{3/2} v_{th}^2} e^{-1/2(1-\gamma/\beta)^{-1} \hat{v}_x^2} \left( e^{-1/2 \hat{v}_y^2} - \gamma e^{-1/2 \beta \hat{v}_y^2} \right), \tag{18}$$

$0 \leq \gamma \leq 1, \quad \beta > 1$.
$f_y^{(0)}$ has a minimum with respect to $v_y$ for $\alpha \beta > 1$. At the same time the system is linearly stable for (see Appendix)

$$\alpha = \beta^{-\epsilon}, \quad \frac{1}{2} \leq \epsilon < 1.$$  \hfill (19)

Table 1 gives for a number of $\epsilon$'s and $\alpha$'s in the range

$$0.5 < \alpha < 0.9, \quad 0.1 \leq \epsilon \leq 0.9$$  \hfill (20)

the exact numerically obtained values $(k_{\min} r_g)_{\text{num}}$ of $k_{\min} r_g$ together with an approximation $(k_{\min} r_g)_{\text{app}}$ of this quantity given by

$$(k_{\min} r_g)_{\text{app}} = 1.13 \sqrt{\frac{f_{\max} + f_{\min}}{2 H_{\max} (f_{\max} - f_{\min})}}$$  \hfill (21)

with

$$H_{\max} = \frac{1}{2} v_{y_{\max}}^2 = \frac{\ln \alpha \beta}{\beta - 1},$$

$$f_{\max} = f_y^{(0)}(v_{y_{\max}}, v_z), \quad f_{\min} = f_y^{(0)}(0, v_z).$$

Relation (21) corresponds to relation (17) for the case that the minimum of $f_y^{(0)}$ is at $\dot{v}_y = 0$, in which case $V_y \approx \Delta \dot{v}_y$ and $\frac{1}{2} (\Delta \dot{v}_y)^2 = H_{\max}$. Table 1 contains for more detailed information also the values of the quantities $\alpha \beta$, $H_{\max}$ and $f_{\max}/f_{\min}$. The main result is that relation (21) or, equivalently, relation (17) provides a reasonable approximation for $k_{\min} r_g$. This is in agreement with the character of the eigenfunctions to the lowest eigenvalue shown in Figs. 1 together with the corresponding distribution functions. As expected, the eigenfunctions are of the form which was assumed for deriving relation (17). The only exception is the rather exotic case $\epsilon = 0.1, \alpha = 0.5$. Table 1 tells us, furthermore, that the localization $1/k_{\min} r_g$ necessary for negative energy waves tends to be stronger for stable plasmas than for unstable ones.

5. Conclusions

Condition (1), which is fulfilled in any magnetically confined plasma, is only obtained with infinitely strongly localized perturbations. Negative energy waves should, however, be dangerous only if their...
Table 1. Parameters for the distribution functions (18):
y = ß~f, see (19); Hmax = ln (ß/ß - 1); value of H = v²/2 at which f_v(v², v) has its maximum; f_max = f_v(vmax, v); f_min = f_v(0, v); (kminr g)app: approximation of kminr given by (21); (kminr g)num: numerical value of kminr.

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<th>ε</th>
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wavelengths are not too small. By treating a magnetized, homogeneous plasma for illustration it is shown via a Hermitian eigenvalue problem that the necessary localization should be generally related to the average gyroradius r_g v of the relevant particle species v. For unstable plasmas the least localization is of the order of r_g, whereas for stable plasmas it is a smaller fraction of r_g v. Since the Vlasov equation is valid only for length scales larger than the Debye length, all such negative energy waves should be physically meaningful if the gyroradius of the relevant species is larger than the Debye length.

Acknowledgement

The author is grateful to W. Kerner and E. Schwarz for the numerical solution of (14).

Appendix

Stability Regime for Waves Propagating Perpendicularly to B(0)

We discuss here waves with their electric field vectors parallel to B(0). In this case the dispersion relation is

$$\frac{\omega_p^2}{c^2} \varepsilon_{33} - k^2 = 0$$  \hspace{1cm} (A1)

with

$$\varepsilon_{33} = 1 + \sum_v \frac{\omega_p^2}{\omega_v^2} \left( -1 + \frac{x}{\omega_v^2} \frac{\partial f_v^{(p)}}{\partial \varepsilon_{33} / \omega_v^2} \sum_{n=1}^\infty \frac{2n^2 \omega_v^2}{\omega^2 - n^2 \omega_v^2} J_n \left( \frac{k v_i}{\omega_v} \right) \right)$$

$$\omega_p^2$$ is the plasma frequency of the species v, J_n(z) the Bessel function of order n, and f_v^{(p)}(v²) is given by

$$f_v^{(p)}(v^2) = 2 \pi \int_{-\infty}^\infty dv_z v_z^2 f_v^{(0)}(v^2, v_z)$$

$$= e^{-1/2 v²} - \frac{1}{\alpha} e^{-1/2 \beta v²}.$$  \hspace{1cm} (A3)

(see (18)). With

$$x = \text{Re}(\omega), \quad y = \text{Im}(\omega)$$  \hspace{1cm} (A4)

we can decompose the dispersion relation (A1), (A2) into its real and imaginary parts:

$$x^2 - y^2 - k^2 c^2 + \sum_v \omega_v^2$$

$$\cdot \left( -1 + 2 \sum_v \omega_p^2 \sum_{n=1}^\infty \frac{n^2 \omega_v^4}{\omega^2 - n^2 \omega_v^2} + 4 x^2 y^2 \right)$$

$$\cdot \sum_v \frac{x}{\omega_v^2} \frac{\partial f_v^{(p)}}{\partial \varepsilon_{33} / \omega_v^2} J_n \left( \frac{k v_i}{\omega_v} \right) = 0, \quad (A5a)$$

The second equation (A5b) can only be fulfilled with x y \neq 0 if for some v and n

$$\int_0^\infty v^2 dv \int_{-\infty}^\infty dv_2 \frac{\partial f_v^{(p)}}{\partial \varepsilon_{33} / \omega_v^2} J_n \left( \frac{k v_i}{\omega_v} \right) > 0.$$

(A6)

With f_v^{(p)} from (A5) one finds

$$\int_0^\infty v^2 dv \int_{-\infty}^\infty dv_2 \frac{\partial f_v^{(p)}}{\partial \varepsilon_{33} / \omega_v^2} J_n \left( \frac{k v_i}{\omega_v} \right) = -e^{-z} I_n(z)$$

$$+ \alpha e^{-z/\beta} I_n(z/\beta),$$

$$z = k^2 v_i^2/\omega_v^2.$$  \hspace{1cm} (A7)

I_n(z) is the modified Bessel function of order n. From the asymptotic behaviour of I_n one infers that x and \beta, related by

$$\beta < 1, \quad (A9)$$
do not allow positive values of the integral (A7) even for non-monotonic distribution functions, i.e. \beta > 1.
The latter is always fulfilled with (A9) for \( \beta > 1 \).

With (A9) we can write (A7) in the form

$$\int_0^\infty v_1 \frac{df}{dv_1} \frac{\partial f}{\partial v_1} J_n^2 \left( \frac{k v_1}{\omega_\nu} \right) = -\frac{1}{\beta^2} (z^2 e^{-z} I_n(z) - (z/\beta)^n e^{-z/\beta} I_n(z/\beta)).$$ \hspace{1cm} (A10)

If we can prove that

$$w_n = x^2 e^{-x} I_n(x)$$ \hspace{1cm} (A11)

is a monotonically increasing function of \( x \), then the integral (A10) is negative and the system is stable with respect to the considered perturbations. From the equation for \( I_n(x) \),

$$x^2 \frac{d^2 I_n}{dx^2} + x \frac{dI_n}{dx} - (x^2 + n^2) I_n = 0,$$ \hspace{1cm} (A12)

one finds for \( w_n \) the equation

$$x^2 w_n'' + 2x (x - \varepsilon + \frac{1}{2}) w_n' - (n^2 + 2x (\varepsilon - \frac{1}{2}) - \varepsilon^2) w_n = 0.$$ \hspace{1cm} (A13)

This proves that (A 9) describes systems that are stable with respect to the perturbations considered.