Stationary Solutions of Incompressible Stokes Equation for a Circular Pipe

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Z. Naturforsch. 42a, 543–546 (1987); received January 23, 1987

Stationary modes of the incompressible Stokes equation are derived using the method of potentials. Their relation to instationary modes is discussed.

In a recent paper [1] a method developed by Hansen, Stratton, Morse, Feshbach [2] ("HSMF-method") for solving linear vector-wave equations in electrodynamics was applied to find analytical expressions for three-dimensional time-dependent solutions of the Stokes equation for a pipe with circular cross-section. In this paper we derive analytical expressions for stationary modes of the stationary Stokes equation

\[
\frac{\partial^2 p}{\partial t^2} = - \nabla \cdot \mathbf{u}, \quad \nabla \times \mathbf{u} = 0 \quad (1)
\]

with the help of "potentials" \(a, b\) for the same geometry and boundary condition as in [1].

Expressing \(u\) and \(p\) by potentials \(a(r, \phi, z), b(r, \phi, z)\) [(1), ref. 24]

\[
\begin{align*}
\mathbf{u} &= \text{rot } a + \text{rot } b, \quad p = p_0 + \frac{\partial q}{\partial r} \text{div } (e_z \Delta b), \\
a &= a e_z, \quad b = b e_z,
\end{align*}
\]

we get for \(a, b\) the differential equations

\[
\Delta a = 0, \quad \Delta b = 0. \quad (3)
\]

Introducing cylindrical coordinates \(r, \phi, z\) and separating \(a(r, \phi, z)\) and \(b(r, \phi, z)\) into \(a(r, \phi, z) = a_i(r) a_j(\phi) a_k(z), \quad b(r, \phi, z) = b_i(r) b_j(\phi) b_k(z)\) we find for \(k \neq 0\) the solutions

\[
\begin{align*}
a(r, \phi, z) &= a_m J_m(ikr) e^{im\phi} e^{ikz}, \\
b(r, \phi, z) &= b_m J_m(ikr) e^{im\phi} e^{ikz} \\
&+ \frac{c_m}{ik} J'_m(ikr) e^{im\phi} e^{ikz},
\end{align*}
\]

which give

\[
\begin{align*}
u_r &= \left[ a_m \frac{im}{r} J_m + b_m k^2 J_{m+1} - \frac{m}{ikr} J_m \right] \\
&+ c_m \left( \frac{m^2}{ikr} - ik r J_m \right) e^{im\phi} e^{ikz}, \\
u_\phi &= \left[ a_m ik \left( J_{m+1} - \frac{m}{ikr} J_m \right) - b_m \frac{mk}{r} J_m \right] \\
&+ c_m \left( \frac{m}{ikr} J_m - J_{m+1} \right) e^{im\phi} e^{ikz}, \\
u_z &= \left[ -b_m k^2 J_m \\
&+ c_m \left( (2 + m) J_m - ikr J_{m+1} \right) \right] e^{im\phi} e^{ikz}. \quad (5)
\end{align*}
\]

The boundary condition \(u(r = R, \phi, z) = 0\) leads to the dispersion relation

\[
D_S(k, m) := ik R J_m^2(k R) \\
+ [k^2 R^2 - 2m(2 + m)] J_{m+1}^2(k R) \quad (6)
\]

For \(m = 0\) we find

\[
J_1 \left[ J_0 + \frac{2i}{k R} J_0 J_1 + J_1^2 \right] = 0, \quad (7)
\]

which means either

\[
\begin{align*}
a) & \quad J_1(i k_{(j)} R) = 0, \quad k_{(j)} = i k_{(j)}, \\
& \quad (k_{(j)} > 0 \text{ for } z > 0), \\
& \quad a_0 = 0, \quad b_0 = c_0 = 0, \quad j = 1, 2, 3, \ldots \quad (7a)
\end{align*}
\]

or

\[
\begin{align*}
b) & \quad J_0^2 + \frac{2i}{k_{(j)} R} J_0 J_1 + J_1^2 = 0 \iff J_0 J_2 = J_1^2, \\
& \quad a_0 = 0, \quad b_0 \neq 0, \quad c_0 \neq 0. \quad (7b)
\end{align*}
\]

See also [3], p. 547.
Solutions \( u(r, \varphi, z) \) which belong to a) or b) are given by

a) \( u^{(0,j)} = -e^{i k^{(j)} z} \),

\[
\begin{align*}
\hat{a}_0^{(j)} &= a_0 k^{(j)} , \quad (8a) \\
u^{(0,j)}_0 &= \frac{2 J_0(i k^{(j)} R)}{2 J_0(i k^{(j)} R) - i k^{(j)} R J_1(i k^{(j)} R)} e^{i k^{(j)} z} , \quad (8b)
\end{align*}
\]

b) \( u^{(0,j)}_\varphi = \hat{b}_0^{(j)} \left[ J_1(i k^{(j)} r) - \frac{r}{R} J_0(i k^{(j)} r) - J_1(i k^{(j)} R) \right] e^{i k^{(j)} z} , \]

\[
\begin{align*}
a_0 &= 0 , \\
c_0 &= \frac{b_0(k^{(j)})^2 J_1(i k^{(j)} R)}{i k^{(j)} R J_0(i k^{(j)} R)}
\end{align*}
\]

Solutions for \( m \neq 0 \) can be found from (5) with

\[
\begin{align*}
c_m^{(j)} &= \hat{b}_m^{(j)} \left( 2 + m \right) J_m(i k^{(j)} R) / \left( 2 J_0(i k^{(j)} R) - i k^{(j)} R J_1(i k^{(j)} R) \right) , \\
d_m^{(j)} &= \hat{b}_m^{(j)} i R / m J_m(i k^{(j)} R) \left( 2 + m \right) J_m(i k^{(j)} R) / \left( 2 J_0(i k^{(j)} R) - i k^{(j)} R J_1(i k^{(j)} R) \right) ,
\end{align*}
\]

where \( k^{(j)} \) are solutions of (6) and \( \hat{b}_m^{(j)} = b_m(k^{(j)})^2 \).

The stationary pressure functions \( p^{(m,j)}(r, \varphi, z) \) for \( k^{(j)} \neq 0 \), which follow from (2), (4) are given by

\[
\begin{align*}
p^{(m,j)}(r, \varphi, z) &= p_0 - 2q_0 v_0 i k^{(j)} c_m^{(j)} J_m(i k^{(j)} r) e^{i m \varphi} e^{i k^{(j)} z} , \\
J_m(i k^{(j)} R) &= -J_{m+1}(i k^{(j)} R) + \frac{m}{i k^{(j)} R} J_m(i k^{(j)} R) = 0 , m > 0 ,
\end{align*}
\]

as might be supposed from Figs. 1, 2 in [1].

The analytical limit process \( \sigma \to 0 \) applied to (19) of [1], where \( \sigma \) is the time-separation constant, does not give the dispersion relation for the stationary modes, since \( \lim_{\sigma \to 0} D(k, m, \sigma) = 0 \). \( D(k, m, \sigma) \) denotes the right hand side of equation (19) in [1] for incompressible flow. Instead, the following relation holds \((k \neq 0)\):

\[
D_S(k, m) = \text{const} \left( \frac{\partial D(k, m, \sigma)}{\partial \sigma} \right) _{\sigma = 0} .
\]

As can be shown by simple but lengthy calculations a similar relation can be found between the stationary modes \( u^{(m,j)}(r, \varphi, z) \) and the instationary solutions \( u^{(m)}(r, \varphi, z, t, k, \sigma) \) (equation (17) in [1]) which fulfill the boundary condition \( u^{(m)}(r = R, \varphi, z, t, k, \sigma) = 0 \):

\[
u^{(m,j)}(r, \varphi, z, t, k, \sigma) = \left. \frac{\partial u^{(m)}(r, \varphi, z, t, k, \sigma)}{\partial \sigma} \right|_{\sigma = 0} ,
\]

\[
A(m, k^{(j)}) (a_0 = 0 \text{ for } m = 0) .
\]

\( A(m, k^{(j)}) \) are constants and \( k^{(j)} \) is a solution of (6). The only exception of relation (12) is the stationary solution for \( m = 0 \), \( a_0 = 0 \), \( b_0 = 0 \), \( c_0 = 0 \). In this special case \( u^{(0,j)}(r, z) \) follows directly from

\[
u^{(0,j)}(r, z, t, k, \sigma) = -e^{i k^{(j)} z} \left( \frac{\sigma}{v_0} - k^2 \right) J_0(\sqrt{\frac{\sigma}{v_0} - k^2}) e^{-i \sigma t} ,
\]

for \( \sigma = 0 \) since \( u^{(0,j)}(r, z, t, k, \sigma = 0) \neq 0 \). For \( \sigma = 0 \) \((a_0 = 0 \text{ for } m = 0)\) one gets

\[
\lim_{\sigma \to 0} u^{(m)}(r, \varphi, z, t, k, \sigma) = 0 .
\]
(Eq. (16) of [1] for \( c = \infty \)) is found to be
\[
p^{(m,j)}(r, \varphi, z) = p_0 + \frac{\delta}{d \sigma} \left( \frac{p^{(m,j)}(r, \varphi, z; k, \sigma)}{d \sigma} \right)_{\sigma = 0} = 0
\]
\[
\cdot B(m, k^{(j)}) ,
\]
(14)
where \( B(m, k^{(j)}) \) are constants. Without explicit knowledge of the stationary functions \( D_S(k, m) \), \( u^{(m,j)}(m, z) \), it does not seem possible to state the relations (11), (12), (14). In order to get the relations (12) and (14), the process \( a \to 0 \) with \( b_m = \text{const} \) must be used, where the constants \( b_m \) enter the functions \( u^{(m,j)}(m, \varphi) \) according to equation (18) of [1] for \( c = \infty \). If one interprets the functions \( w^{(m,j)}(m, \varphi) \) or \( u^{(m,j)}(m, \varphi) \) for \( k \neq 0 \) as flow patterns in the pipe, the corresponding Reynolds-numbers must be assumed to be very small, which means that the coefficients \( a_0, b_m \) must be sufficiently small.

The case of vanishing separation constant \( k \) is of certain interest. If one solves (1) to (3) for \( k = 0 \) and \( u(r = R, \varphi, z) = 0 \), only the trivial solution \( u \equiv 0 \), \( p = \text{const} \) results: especially the Hagen-Poiseuille-solution for \( m = 0 \) does not appear. The reason is that the Hagen-Poiseuille-solution \( u_{HP}(r) = \frac{z / 0 (1 - r^2/R^2)}{0} \) cannot be represented by (3). For \( k = 0, m = 0 \), however, (1) is solved by
\[
\begin{align*}
\mathbf{u}^{(0,0)}(r) &= \text{rot} \cdot \text{rot} b_r^{(0,0)}(r), \\
p^{(0,0)}(r) &= p_0 - c \, \varphi_0 \, v_0 \, z, \\
b_r^{(0,0)}(r) &= \varphi, b_z^{(0,0)}(r) = c, \\
A_r &= \frac{1}{r} \left( r \, \frac{d}{dr} \right),
\end{align*}
\]
where \( c \) is an arbitrary constant. The solution of \( A_r, A_z, b_r^{(0,0)}(r) = c \) consists of the general solution of the homogeneous equation and a special solution of the inhomogeneous equation. Taking into account the non-trivial, non-singular terms only, we have
\[
b_r^{(0,0)}(r) = b_0 r^2 + \frac{c}{64} r^4 + b_1 r^2 \ln r ,
\]
\[
A_r b_r^{(0,0)}(r) = \frac{c r^2}{4} + 4 b_0 
\quad \quad \quad \quad + 4 b_1 (\ln r + 1), \quad b_1 = 0 ,
\]
(16)
which gives
\[
\mathbf{u}^{(0,0)}(r) = \frac{c}{4 R^2} \left( 1 - \frac{r^2}{R^2} \right) \varphi , \quad \mathbf{P}^{(0,0)} = p_0 - c \, \varphi_0 \, v_0 \, z .
\]
As already Sexl [3] observed, the Hagen-Poiseuille solution can be found also from the time-dependent solutions of the Stokes equation for \( m = 0, k = 0 \) by the process \( \lim_{\sigma \to 0} u^{(0)}(r, t, \sigma) \), where \( u^{(0)} \) denotes the functions found by Sexl ([3], p. 575). In order to find all stationary solutions for \( k = 0 \) from the time-dependent solutions of the Stokes equation by using the HSMF-method we take the solutions found by Brosa [1]. Since he did not separate the Helmholtz-equation for \( k = 0 \) quite correctly, we summarize the three types of time-dependent solutions for \( k = 0 \):

a) \( m = 0, \varphi_0 \) arbitrary, \( b_0 \) arbitrary:
\[
J_1 \left( \sqrt{\frac{\sigma_0}{v_0}} R \right) = 0 , \quad \sigma = 1, 2, 3, \ldots , \quad u_r^{(0)} = 0 , \quad u_\varphi^{(0)} = 0 , \quad u_z^{(0)} = b_0 \varphi_0 J_0 \left( \sqrt{\frac{\sigma_0}{v_0}} R \right) \left( 1 - \frac{r^2}{R^2} \right) e^{\sigma_0 \sigma \sigma} .
\]

b) \( m = 0, b_0 \) arbitrary, \( \varphi_0 \) arbitrary:
\[
\begin{align*}
\mathbf{u}^{(0)} &= b_0 \sigma_0 \varphi_J \left[ J_0 \left( \sqrt{\frac{\sigma_0}{v_0}} R \right) \right] \\
\mathbf{u}^{(0)} &= b_0 \sigma_0 \varphi_J \left[ J_0 \left( \sqrt{\frac{\sigma_0}{v_0}} R \right) \right] e^{\sigma_0 \sigma \sigma} .
\end{align*}
\]

c) \( m = 0, 1, 2, \ldots , b_m \) arbitrary:
\[
\begin{align*}
J_m \left( \sqrt{\frac{\sigma_m}{v_0}} R \right) &= 0 , \quad u_\varphi^{(m)} = 0 , \quad u_\varphi^{(m)} = 0 , \quad u_z^{(m)} = b_m \sigma_0 \varphi_J J_m \left( \sqrt{\frac{\sigma_m}{v_0}} R \right) e^{\sigma_0 \sigma \sigma} .
\end{align*}
\]
If we look for stationary solutions with \( k = 0 \) as result of a process \( \sigma \to 0 \), \( b_m = \text{const} \), only solutions of type b) can be used since in the other two cases \( \sigma = 0 \) cannot be approached smoothly. Making explicit the argument used in [1] we get for
\[
\begin{align*}
\mathbf{u}^{(0)}(r, t, \sigma)
\end{align*}
\]
\[
= b_0 \sigma_0 \varphi_0 J_0 \left( \sqrt{\frac{\sigma_0}{v_0}} R \right) - J_0 \left( \sqrt{\frac{\sigma_0}{v_0}} R \right) e^{\sigma_0 \sigma \sigma} .
\]
the relations
\[ u_0^{(0)}(r, t, \sigma = 0) = 0, \quad \left( \frac{\partial u_0^{(0)}(\sigma)}{\partial \sigma} \right)_{\sigma = 0} = 0, \]
\[ \left( \frac{\partial^2 u_0^{(0)}(\sigma)}{\partial \sigma^2} \right)_{\sigma = 0} = \frac{b_0}{2v_0} \left( R^2 - r^2 \right). \]

Thus we find
\[ \lim_{\sigma \to 0} u_0^{(0)}(r, t, \sigma) = \frac{b_0}{4v_0} \left( R^2 - r^2 \right) e_z =: u_{\text{HP}}(r), \]
\[ u_0 = b_0^* \frac{R^2}{4v_0}. \]

In the same way the stationary pressure \( p_{\text{HP}}(z) \) follows from
\[ p_{\text{HP}}(z) = \lim_{\sigma \to 0} p^{(0)}(z, t, \sigma), \]
where \( p^{(0)}(z, t, \sigma) \) is given by ([1], Eq. (27))
\[ p^{(0)}(z, t, \sigma) = p_0 - \frac{\sigma}{v_0} b_0 \frac{\sigma}{v_0} J_0 \left( \sqrt{\frac{\sigma}{v_0}} R \right) z e^{-\sigma t}. \]

With
\[ p^{(0)}(z, t, \sigma = 0) = p_0, \quad \left( \frac{\partial p^{(0)}(\sigma)}{\partial \sigma} \right)_{\sigma = 0} = 0, \]
\[ \left( \frac{\partial^2 p^{(0)}(\sigma)}{\partial \sigma^2} \right)_{\sigma = 0} = -\frac{2}{v_0} b_0 \frac{R^2}{R^2} z. \]

we get finally
\[ p_{\text{HP}}(z) = p_0 - \frac{\sigma}{v_0} b_0^* z = p_0 - 4 \frac{\sigma}{v_0} \frac{u_0}{R^2} z. \]

Contrary to the stationary solutions \( u^{(m,0)}(r, \varphi, z) \), the coefficient \( b_0^* \) (or \( u_0 \)) which appears in \( u_{\text{HP}}(r) \) may take values which give large Reynolds-numbers since \( u_{\text{HP}}(r) \) solves the stationary non-linear Navier-Stokes equation.

With regard to the infinite pipe the Hagen-Poiseuille-solution remains as the only stationary solution of the Stokes equation which fulfills the (incomplete) boundary condition \( u(r = R) = 0 \), which possesses continuous partial derivatives everywhere and which is bounded for \(-\infty < z < \infty\).

The functions \( u^{(m,0)}(r, \varphi, z) \) with \( k \neq 0 \) represent solutions of the Stokes equation for the semi-infinite pipe with \( u^{(m,0)}(r, \varphi, z \to \infty) = 0 \) and a prescribed boundary condition at \( z = 0 \).

We want to mention that also \( \Delta a = 0 \) (2) is only sufficient to solve (1). If for the case \( m = 0, k = 0 \) one lets \( \Delta a = c \), one finds an additional solution \( u(r) = -\frac{1}{c} r e_\varphi \), which gives no contribution for homogeneous boundary conditions \( u^{(0,0)}(r) = 0 \), but for a boundary condition
\[ u^{(0,0)}(r = R) = u^{(0,0)}(r = R) = 0, \]
\[ u_\varphi^{(0,0)}(r = R) = u_\varphi^{(0,0)}, \]
which characterizes a rotating pipe.