Stationary Solutions of Incompressible Stokes Equation for a Circular Pipe

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Stationary modes of the incompressible Stokes equation are derived using the method of potentials. Their relation to instationary modes is discussed.

In a recent paper [1] a method developed by Hansen, Stratton, Morse, Feshbach [2] ("HSMF-method") for solving linear vector-wave equations in electrodynamics was applied to find analytical expressions for three-dimensional time-dependent solutions of the Stokes equation for a pipe with circular cross-section. In this paper we derive analytical expressions for stationary modes of the stationary Stokes equation

\[
\begin{align*}
\frac{1}{\rho_0} \frac{\partial \rho}{\partial \tau} &= - \nu_0 \text{rot rot } \mathbf{u}, \quad \text{div } \mathbf{u} = 0 \\
&= - \nu_0 \text{rot rot } \mathbf{u}, \quad \text{div } \mathbf{u} = 0 \quad (1)
\end{align*}
\]

with the help of "potentials" \(a, b\) for the same geometry and boundary condition as in [1].

Expressing \(\mathbf{u}\) and \(p\) by potentials \(a(r, \tau), b(r, \tau)\) ([1], ref. 24)
\[
\begin{align*}
\mathbf{u} &= \text{rot } a + \text{rot } b, \quad p = p_0 + \rho_0 \nu_0 \text{div } (e_z \nabla b), \\
a &= a e_z, \quad b = b e_z, \\
\end{align*}
\]
we get for \(a, b\) the differential equations
\[
\begin{align*}
\Delta a &= 0, \quad \Delta b = 0. \quad (3)
\end{align*}
\]

Introducing cylindrical coordinates \(r, \varphi, z\) and separating \(a(r, \varphi, z) = a_r(r) a_\varphi(\varphi) a_z(z)\) \(b(r, \varphi, z) = b_r(r) b_\varphi(\varphi) b_z(z)\) we find for \(k \neq 0\) the solutions
\[
\begin{align*}
a(r, \varphi, z) &= a_m J_m(ikr) e^{im\varphi} e^{ikz}, \\
b(r, \varphi, z) &= b_m J_m(ikr) e^{im\varphi} e^{ikz} \\
&+ \frac{c_m r}{ik} J'_m(ikr) e^{im\varphi} e^{ikz}, \quad (4)
\end{align*}
\]

which give
\[
\begin{align*}
\mathbf{u}_r &= \left[ a_m \frac{i m}{r} J_m + b_m k^2 \left( J_{m+1} - \frac{m}{ikr} J_m \right) \\
&+ c_m \left( \frac{m^2}{ikr} - ikr \right) J_m \right] e^{im\varphi} e^{ikz}, \\
\mathbf{u}_\varphi &= \left[ a_m ik \left( J_{m+1} - \frac{m}{ikr} J_m \right) - b_m \frac{mk}{r} J_m \\
&+ c_m im \left( \frac{m}{ikr} J_m - J_{m+1} \right) \right] e^{im\varphi} e^{ikz}, \\
\mathbf{u}_z &= [-b_m k^2 J_m \\
&+ c_m ((2 + m) J_m - ikr J_{m+1})] e^{im\varphi} e^{ikz}. \quad (5)
\end{align*}
\]

The boundary condition \(\mathbf{u}(r = R, \varphi, z) = 0\) leads to the dispersion relation
\[
D_S(k, m) := ik R m J_m^2 (ik R) \\
+ [k^2 R^2 - 2m(2 + m)] J_m^2 J_{m+1} \\
+ i k R (2 + 3m) J_m J_{m+1}^2 \\
+ k^2 R^2 J_{m+1} = 0, \quad \quad (6)
\]

For \(m = 0\) we find
\[
J_1 \left[ J_0^2 + \frac{2i}{k R} J_0 J_1 + J_1^2 \right] = 0, \quad (7)
\]

which means either
\[
\begin{align*}
a) \quad J_1 (ik^{\jmath} R) &= 0, \quad k^{\jmath} = ik^{\jmath}, \\
(k^{\jmath} > 0 & \text{ for } z > 0), \\
a_0 &\neq 0, \quad b_0 = c_0 = 0, \quad \jmath = 1, 2, 3, \ldots \quad (7a)
\end{align*}
\]
or
\[
\begin{align*}
b) \quad J_0^2 + \frac{2i}{k^{\jmath} R} J_0 J_1 + J_1^2 &= 0 \iff J_0 J_2 = J_1^2, \quad (7b)
\end{align*}
\]
\[
\begin{align*}
a_0 &= 0, \quad b_0 \neq 0, \quad c_0 \neq 0 .
\end{align*}
\]

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Solutions $u(r, \varphi, z)$ which belong to a) or b) are given by

a) $u^{(0,j)} = - e^{i k^0 z} \cdot \Phi^0_0(k^0 r) e^{i m \varphi}$

\[ a^0_j = a_0 k^0_j, \quad (8a) \]

b) $u^{(0,0)} = \Phi^0_0(k^0 r) \cdot J_0(k^0 r)$

\[ u^{(0,0)} = 0, \quad (8b) \]

Solutions for $m \neq 0$ can be found from (5) with

\[ c_m(k^0) = \Phi_m(k^0) \cdot \frac{1}{2 + m} J_m(k^0 R) \cdot \frac{1}{J_m(k^0 R)} \cdot (2 + m) J_m(k^0 R) - i k^0 R J_m(k^0 R) \cdot (9) \]

where $k^0$ are solutions of (6) and $\Phi_m(k^0) = b_m(k^0)^2$. The stationary pressure functions $p^{(m,j)}(r, \varphi, z)$ for $k^0 \neq 0$, which follow from (2), (4) are given by

\[ p^{(m,j)}(r, \varphi, z) = p_0 - 2 q_0 v_0 i k^0 c_m(k^0) \cdot J_m(k^0 r) e^{i m \varphi} e^{i k^0 z} \cdot \Phi^0_0(k^0 r). \quad (10) \]

If $k^0$ is a solution of (6) the same holds for $-k^0$. Numerical calculations for $m = 0$ ([1]) with small but finite values of $\sigma$ are in agreement with (7 a) to (8 b). As can be seen from (6), there exist no solutions for $k^0$ with

\[ J_m(k^0 R) - J_{m+1}(k^0 R) \cdot (0, m > 0), \]

as might be supposed from Figs. 1, 2 in [1].

The analytical limit process $\sigma \to 0$ applied to (19) of [1], where $\sigma$ is the time-separation constant, does not give the dispersion relation for the stationary modes, since $\lim_{\sigma \to 0} D(k, m, \sigma) = 0$. $D(k, m, \sigma)$ denotes the right hand side of equation (19) in [1] for incompressible flow. Instead, the following relation holds ($k \neq 0$):

\[ D(k, m) = \text{const} \cdot \frac{\partial D(k, m, \sigma)}{\partial \sigma} \bigg|_{\sigma = 0}. \quad (11) \]

As can be shown by simple but lengthy calculations a similar relation can be found between the stationary modes $u^{(m,j)}(r, \varphi, z)$ and the instationary solutions $u^{(m)}(r, \varphi, z, t, k, \sigma)$ (equation (17) in [1]) which fulfill the boundary condition $u^{(m)}(r = R, \varphi, z, t, k, \sigma) = 0$.

\[ u^{(0,0)}(r, \varphi, z, t, k, \sigma) = - e^{i k^0 z} \cdot \Phi^0_0(k^0 r) e^{i m \varphi} \cdot \Phi^0_0(k^0 r). \quad (13) \]

for $\sigma = 0$ since $u^{(0,0)}(r, \varphi, z, t, k, \sigma) = 0$. For $\sigma = 0$ ($a_0 = 0$ for $m = 0$) one gets

\[ \lim_{\sigma \to 0} u^{(m)}(r, \varphi, z, t, k, \sigma) = 0. \]

The relation between the stationary pressure functions (10) and the corresponding instationary ones
(Eq. (16) of [1] for $c_r = \infty$) is found to be
\[ p^{(m,j)}(r, \varphi, z) = p_0 + \frac{\partial}{\partial \sigma} \left[ p^{(m,j)}(r, \varphi, z; k, \sigma) e^{\sigma \tau} \right]_{\sigma = 0} \cdot B(m, k^{(j)}) , \tag{14} \]
where $B(m, k^{(j)})$ are constants. Without explicit knowledge of the stationary functions $D_S(k, m)$, $u^{(m,j)}(r, \varphi, z)$, $p^{(m,j)}(r, \varphi, z)$ it does not seem possible to state the relations (11), (12), (14). In order to get the relations (12) and (14), the process $a \to 0$ with $b_m = b_m^* = \text{const}$ must be used, where the constants $b_m$ enter the functions $u^{(m,j)}$, $u^{(m)}$ according to equation (18) of [1] for $c = \infty$. If one interprets the functions $u^{(m,j)}$, $u^{(m)}$ for $k \neq 0$ as flow patterns in the pipe, the corresponding Reynolds-numbers must be assumed to be very small, which means that the coefficients $a_0$, $b_m$ must be sufficiently small.

The case of vanishing separation constant $k$ is of certain interest. If one solves (1) to (3) for $k = 0$ and $u(r = R, \varphi, z) = 0$, only the trivial solution $u \equiv 0$, $p = \text{const}$ results: especially the Hagen-Poiseuille-solution for $m = 0$ does not appear. The reason is that the Hagen-Poiseuille-solution $u_{HP}(r) = \frac{z}{\eta} \left( 1 - r^2/R^2 \right) e_z$ cannot be represented by (3). For $k = 0, m = 0$, however, (1) is solved by
\[ u^{(0,0)}(r) = \text{rot} \cdot \text{rot} b^{(0,0)}_r(r) , \quad p^{(0,0)}(r) = p_0 - c \tau_0 V_0 (R^2) , \quad b^{(0,0)}_r = \frac{1}{r} \left( r \frac{d}{dr} \right) , \tag{15} \]
where $c$ is an arbitrary constant. The solution of $A_r b^{(0,0)}_r(c) = c$ consists of the general solution of the homogeneous equation and a special solution of the inhomogeneous equation. Taking into account the non-trivial, non-singular terms only, we have
\[ b^{(0,0)}_r(r) = b_0 r^2 + \frac{c}{64} r^4 + b_1 r^2 \ln r , \]
\[ A_r b^{(0,0)}_r = \frac{c r^2}{4} + 4 b_0 + 4 b_1 \ln(r + 1) , \quad b_1 = 0 , \tag{16} \]
which gives
\[ u^{(0,0)}(r, \varphi, z) = \frac{c}{4 R^2} \left( 1 - \frac{r^2}{R^2} \right) e_z , \quad p^{(0,0)}(r, \varphi, z) = p_0 - c \tau_0 V_0 (R^2) . \tag{17} \]
As already Sexl [3] observed, the Hagen-Poiseuille solution can be found also from the time-dependent solutions of the Stokes equation for $m = 0, k = 0$ by the process $\lim_{c = \text{const}} u^{(0)}_r(r, t, \tau), u^{(0)}_z = 0$. If one interprets the functions $u^{(0)}_r, u^{(0)}_z$ for $k \neq 0$ as flow patterns in the pipe, the corresponding Reynolds-numbers must be assumed to be very small, which means that the coefficients $a_0$, $b_m$ must be sufficiently small.

When we look for stationary solutions with $k = 0$ as result of a process $a \to 0, c_r = \text{const}$, only solutions of type b) can be used since in the other two cases $\tau = 0$ cannot be approached smoothly. Making explicit the argument used in [1] we get for
\[ u^{(0)}_z(r, t, \tau) = \frac{c_0}{V_0} \left( J_0 \left( \frac{V}{V_0} r \right) - J_0 \left( \frac{V}{V_0} R \right) \right) e^{\tau \tau} . \tag{18} \]
the relations
\[ u_t^{(0)}(r, t, \sigma = 0) = 0 \], \( \frac{\partial u_t^{(0)}(\sigma)}{\partial \sigma} \bigg|_{\sigma = 0} = 0 \),
\[ \frac{\partial^2 u_t^{(0)}(\sigma)}{\partial \sigma^2} \bigg|_{\sigma = 0} = \frac{b_0}{2 v_0} (R^2 - r^2) \).

Thus we find
\[ \lim_{\sigma \to 0} u_t^{(0)}(r, t, \sigma) = \frac{b_0}{4 v_0} (R^2 - r^2) e_z \]
\[ u_0 = b_0^* R^2 \frac{R}{v_0} \]

In the same way the stationary pressure \( p_{hp}(z) \) follows from
\[ p_{hp}(z) = \lim_{\sigma \to 0} \frac{p^{(0)}(z, t, \sigma)}{b_0 \sigma^2 + b_0^*} \],
where \( p^{(0)}(z, t, \sigma) \) is given by ([1], Eq. (27))
\[ p^{(0)}(z, t, \sigma) = p_0 - \frac{\sigma}{\sqrt{\frac{\sigma}{v_0}}} \frac{\sqrt{\frac{\sigma}{v_0}}}{R} \int e^{-\sigma t} \]
With
\[ p^{(0)}(z, t, \sigma = 0) = p_0 \], \( \left( \frac{\partial p^{(0)}(\sigma)}{\partial \sigma} \right) \bigg|_{\sigma = 0} = 0 \),
\[ \left( \frac{\partial^2 p^{(0)}(\sigma)}{\partial \sigma^2} \right) \bigg|_{\sigma = 0} = -\frac{2 \sigma}{\sqrt{\frac{\sigma}{v_0}}} \]
we get finally
\[ p_{hp}(z) = p_0 - \frac{\sigma}{v_0} b_0^* z = p_0 - 4 \frac{\sigma}{v_0} \frac{u_0}{R^2} z \]

Contrary to the stationary solutions \( u^{(m, j)}(r, \varphi, z) \), the coefficient \( b_0^* \) (or \( u_0 \)) which appears in \( u_{hp}(r) \) may take values which give large Reynolds-numbers since \( u_{hp}(r) \) solves the stationary non-linear Navier-Stokes equation.

With regard to the infinite pipe the Hagen-Poiseuille-solution remains as the only stationary solution of the Stokes equation which fulfills the (incomplete) boundary condition \( u(r = R) = 0 \), which possesses continuous partial derivatives everywhere and which is bounded for \(-\infty < z < \infty \).

The functions \( u^{(m, j)}(r, \varphi, z) \) with \( k \neq 0 \) represent solutions of the Stokes equation for the semi-infinite pipe with \( u^{(m, j)}(r, \varphi, z \to \infty) = 0 \) and a prescribed boundary condition at \( z = 0 \). We want to mention that also \( \Delta a = 0 \) (2) is only sufficient to solve (1). If for the case \( m = 0, k = 0 \) one lets \( \Delta a = c \), one finds an additional solution \( u(r) = -\frac{1}{c} r e_\varphi \), which gives no contribution for homogeneous boundary conditions \( u^{(0, 0)}(r) = 0 \), but for a boundary condition
\[ u^{(0, 0)}(r = R) = u^{(0, 0)}(r = R) = 0 \],
\[ u^{(0, 0)}(r = R) = u^{(0, 0)}(r = R) \]
which characterizes a rotating pipe.