Some Reductive Methods for Nonlinear Waves
in an Elastic Cylinder and a Channel Containing
a Two-Layer Fluid

K. Murawski
Department of Physics AR, Akademicka 13, 20-033 Lublin, Poland
Z. Naturforsch. 41a, 1186 – 1194 (1986); received March 25, 1986

A nonlinear wave theory is developed to review the reductive Taniuti-Wei, the derivative expansion, and the rays methods. Model equations describing wave propagation in a fluid-filled cylinder with a thin wall of elastic rings and a channel containing a two-layer fluid are derived via these methods.

1. Introduction

The study of pressure waves in tubes has a long history. Elastic tubes containing a fluid have been considered for the first time by Thomas Young [1] in connection with modelling the propagation of the arterial pressure. Moodie et al. [2] have recently discussed this problem assuming the fluid to be inviscid. A one-dimensional theory has been extracted by averaging the quantities over the tube cross section. The linear tube has been modelled on the basis of a viscoelastic shell theory which neglects bending moments and shear deformation of the wall. The problem was solved in the form of a Laplace integral which was inverted numerically. The results were tested against experimentally generated pressure pulses. The linear model has been developed to describe the motion and reflection of pressure waves in joined, initially uniform viscoelastic tubes and their subsequent interaction with various junctions characteristic of the arterial system [3]. Wave propagation and shock formation in elastic and viscoelastic fluid-filled tubes for a Mooney-Rivlin material have been discussed by Tait et al. [4]. Also the radial motion of a viscoelastic and thin-walled tube has been introduced by considering a model exhibiting impact and long time relaxed elastic behaviour [5]. A two-dimensional analysis was employed to study linear pulse propagation in thin-walled circularly cylindrical elastic tubes containing an inviscid and incompressible liquid. The results showed that the axial velocity can vary as much as 15 to 20 percent across a cross section [6]. A viscoelastic shell model for transient pressure perturbations in linear fluid-filled tubes has been presented and tested against experiments involving a water-filled latex tube. The experimental and numerical results are in good agreement over the major part of the wave [7].

In 1980 Lamb [8] showed, using Taniuti-Wei method [9], that in the absence of energy dissipation the fundamental set of nonlinear equations describing the irrotational motion of pressure waves in an incompressible liquid that is confined within an infinitely long circular cylinder with thin walls of elastic rings leads to the Korteweg-de Vries equation. For this case the same equation has been derived via Whitham’s method [10] by Murawski [11]. On the contrary, when dissipation is taken into consideration the Burgers equation has been obtained [12].

Waves in a two-layer fluid have been considered by Kakutani et al. [13] and Funakoshi [14] as a simplest model of a stratified shear flow [15]. There may exist two modes which differ in the magnitude of their phase velocities [16]. For the fast mode, the surface and interface waves are in phase, and the amplitude of the surface wave is always larger than that of the interface one. On the other hand, for the slow mode these waves are always 180° out of phase [13]. We develop here a method presented by Shen and Zhong [17] in connection with water waves in a channel containing one layer of fluid.

The paper is arranged as follows: In the next section the fundamental set of equations describing

Reprint requests to K. Murawski, School of Physical Sciences, The Flinders University of South Australia, Bedford Park, South Australia 5042.

0340-4811 / 86 / 1000-1186 $ 01.30/0. – Please order a reprint rather than making your own copy.
waves in an elastic cylinder is presented. Sections 3 and 4 deal with the derivation of the Korteweg-de Vries and the nonlinear Schrödinger equations via the reductive Taniuti-Wei [9, 18] and the derivative expansion [19] methods. Section 5 presents the fundamental set of equations for two-fluid waves in a channel. The Korteweg-de Vries equation with varying coefficients is obtained in Sect. 6 via Shen's method [17]. For a critical density ratio (the nonlinear coefficient is equal to zero) the modified Korteweg-de Vries equation is derived in the same section. Near the critical density ratio the above equations are modified to combine them. Phase diagrams for this equation are presented in the Appendix. Section 7 is devoted to the concluding remarks.

2. Equations of Motion for Nonlinear Waves in a Cylinder

In Sects. 2–4 all investigations deal with one-dimensional irrotational incompressible fluid waves of characteristic amplitude \( l \) and characteristic length \( \lambda \) in an infinitely long tube with thin walls of elastic rings (undisturbed diameter \( 2a \)) to take into account dissipation of energy, nonlinearity and dispersion of the medium on the assumption that \( l < 2a \ll \lambda \). Axial motions of the wall and bending moments are ignored. Then the fundamental set of equations may be written as

\[
A_t + (AV)_x = 0, \tag{2.1}
\]

\[
V_t + VV_x + \frac{1}{\varrho} p_x = 0, \tag{2.2}
\]

\[
A_{tt} + \frac{E}{\varrho_m^2} A - \frac{2\pi a}{\varrho_m h} p = \frac{\pi(\varrho h - 2a q)}{\varrho_m h}, \tag{2.3}
\]

where \( \varrho \) = constant liquid density, \( A \) = area of the cross section, \( V \) = liquid velocity, \( a \) = tube radius at the undisturbed uniform state, \( \varrho_m \) = density of the tube material, \( E \) = Young's modulus in the circumferential direction, \( p \) = liquid pressure, \( q \) = outside pressure, \( h \) = thickness of a tube wall. The subscripts \( x \) and \( t \) denote partial differentiation.

The model equation governing the motion of the tube wall as a linear viscoelastic solid characterized by its relaxation time was that employed in [20], but for our purpose (2.3) is used and may describe impulse propagation in an arterial or nervous system. The experimental results obtained by Greenwald et al. [21] showed close agreement between the synthesised and measured waveforms. A problem of impulse propagation in nervous system was treated in [22].

By introducing the new coordinates

\[
A \to \frac{1}{\pi a^2} A, \quad p \to \frac{2a}{E h} p, \quad V \to \sqrt{\frac{2a q}{E h}} V,
\]

\[
t \to \frac{1}{a} \sqrt{\frac{E}{\varrho_m}} t, \quad x \to \frac{2q}{\varrho_m a h} x, \tag{2.4}
\]

dimensionless equations may be obtained, i.e.,

\[
A_t + (AV)_x = 0, \tag{2.5}
\]

\[
V_t + VV_x + A_x + A_{xtt} = 0. \tag{2.6}
\]

We define two dimensionless small parameters, namely

\[
e = \frac{2a}{\lambda}, \quad \delta = \frac{l}{2a}. \tag{2.7}
\]

\( e \) and \( \delta \) measure the weakness of dispersion and nonlinearity, respectively. Various model equations may be derived depending on the relative magnitudes of these parameters. For example, the Korteweg-de Vries equation is obtained on the assumption that \( \delta = e^2 \) and the nonlinear Schrödinger equation corresponds to \( \delta = e \).

3. The Derivative Expansion Method for the Korteweg-de Vries Equation

As is mentioned in the Introduction, the fundamental set of equations (2.1)–(2.3) has been reduced for the first time to the Korteweg-de Vries equation by Lamb [8] using the reductive method. Also, the Lagrangian for the set of equations (2.5) and (2.6) was found and written as [11]

\[
L = \frac{1}{2} A \psi_t^2 + A \psi_t - \frac{1}{2} A_t^2 + \frac{1}{2} A^2 - A + \frac{1}{2}, \quad V \equiv \psi_e. \tag{3.1}
\]

The Euler-Lagrange equations lead to the Korteweg-de Vries equation.

Our purpose now is to show that a derivative expansion method leads to the same equation. We deal with small amplitudes and long waves. We
introduce the multiple spatial and temporal scales
\[ t_n = \varepsilon^n t, \quad x_n = \varepsilon^n x, \quad n = 1, 2, \ldots \]
(3.2)
expand the dependent variables \( V \) and \( A \) around the undisturbed uniform state,
\[ V = \delta V_1 + \delta^2 V_2 + \ldots, \quad A = 1 + \delta A_1 + \delta^2 A_2 + \ldots \]
(3.3)
and assume that \( \delta = \varepsilon^2 \). On substituting (3.2) and (3.3) into (2.5) and (2.6) we get a sequence of equations by equating the coefficients of like powers of \( \varepsilon \). From the equations for \( \varepsilon^2 \) and \( \varepsilon^3 \) we find
\[ V_1(x_1, x_2, t_1, t_2) = V_1(\zeta_1 t_1, \zeta_2 t_2), \quad A_1 = V_1. \]
(3.4)
The fourth-order equations lead to
\[ V_2 x_1 x_1 - V_2 t_1 + 2A_1 x_1 \xi_1 + 2A_1 x_2 \xi_2 + 3(A_1 A_1) \xi_1 + A_1 x_2 \xi_1 \xi_1 = 0. \]
(3.5)
The second-order terms may be removed if we assume that \( V_2 \) depends on \( x_1 \) and \( t_1 \) through \( \xi_1 \). By transforming to a coordinate system moving with the phase velocity 1, i.e.,
\[ \xi_3 = x_3 - t_3, \quad \tau = t_3, \]
(3.6)
we obtain the Korteweg-de Vries equation
\[ A_1 + \frac{1}{2} A_1 A_1 \xi_1 + \frac{1}{2} A_1 \xi_1 \xi_1 = 0. \]
(3.7)
This equation was modified to contain an integral term including the dissipative effects of viscous boundary layers in the context of water waves [23].

The basic Korteweg-de Vries equation and its solutions were discussed in some detail to obtain \( N \)-soliton [24] and \( N \)-periodical wave [25] solutions. Two-soliton scattering was considered in order to show the shape of the single soliton [26]. The stability of the Korteweg-de Vries waves and solitons was discussed by Infeld [27]. The Korteweg-de Vries equations for water waves were reviewed by Johnson [28].

**4. The Reductive Taniuti-Wei method for the Nonlinear Schrödinger Equation**

In the previous section, we have applied the derivative expansion method to derive the Korte-
weg-de Vries equation for weakly dispersive waves. Here, turning our interest to strongly dispersive waves, we develop the reductive Taniuti-Wei method to obtain the nonlinear Schrödinger equation. For this aim we expand the quantities \( A \) and \( V \) into the series
\[ A = 1 + \sum_{n=1}^{\infty} \varepsilon^n \sum_{m=-\infty}^{\infty} A_m^{(n)}(\zeta, \tau) e^{im(k \xi - \omega \tau)}, \]
(4.1)
\[ V = \sum_{n=1}^{\infty} \varepsilon^n \sum_{m=-\infty}^{\infty} V_m^{(n)}(\zeta, \tau) e^{im(k \xi - \omega \tau)}, \]
where
\[ A_m^{(1)} = V_m^{(1)} = 0, \quad m + \pm 1, \quad A_m^{(1)*} = A_m^{(1)}, \]
(4.2)
\[ \zeta = \varepsilon(x - \lambda t), \quad \tau = \varepsilon^2 t. \]
(4.3)
Substitution of the expansion (4.1) into (2.5) and (2.6) yields for the first-order terms
\[ V_1^{(1)} = \frac{\omega}{k} A_1^{(1)}, \quad \omega^2 = \frac{k^2}{1 + k^2}. \]
(4.4), (4.5)
From the second-order equations we find
\[ i(k V_1^{(2)} - \omega A_1^{(2)}) = \left( \lambda - \frac{\omega}{k} \right) A_1^{(2)} \]
(4.6)
\[ A_1^{(2)} = \frac{1}{2k^2} (A_1^{(1)})^2, \]
(4.7)
\[ V_2^{(2)} = \frac{(1 - 2k^2) \omega}{2k^3} (A_1^{(1)})^2, \]
(4.8)
\[ \lambda = \omega_k. \]
(4.9)
The compatibility condition for the components of \( V_1^{(2)} \) and \( A_1^{(2)} \) can be reduced to the nonlinear Schrödinger equation
\[ i A_1^{(1)} + \frac{\omega^3(12k^6 + 35k^4 + 39k^2 + 9)}{4k^4(4k^2 + 3k^2 + 3)} |A_1^{(1)}|^2 A_1^{(1)} - \frac{3\omega}{2(1 + k^2)^2} A_1^{(1)*} A_1^{(1)} \]
(4.10)
where \( C \) is an integration constant which may be calculated from the boundary conditions.

It is well known that this equation may be solved by means of the inverse scattering transform [16], the direct method [29] and Wronskian determinants [30], and possesses Painlevé test and the Bäcklund transformation [31] to obtain an \( N \)-envelope soliton
solution and a periodic envelope one [32]. The stability of this equation was considered by Infeld and Rowlands [33].

5. The Fundamental Set of Equations for Water Waves in a Channel Containing a Two-Layer Fluid

We consider the motion of an inviscid, incompressible two-layer fluid of constant densities under gravity in an open channel with the bottom varying slowly in the longitudinal direction (Figure 1). We assume that the transverse velocities are much smaller than the longitudinal ones and develop the method presented by Shen and Zhong [17]. Our starting point is the following set of fundamental equations:

Upper layer:

\[ u_{1x} + v_{1y} + w_{1z} = 0, \quad (5.1) \]
\[ u_{1t} + u_1 u_{1x} + v_1 u_{1y} + w_1 u_{1z} + \frac{1}{\rho_1} p_{1x} = 0, \quad (5.2) \]
\[ v_{1t} + u_1 v_{1x} + v_1 v_{1y} + w_1 v_{1z} + \frac{1}{\rho_1} p_{1y} = 0, \quad (5.3) \]
\[ w_{1t} + u_1 w_{1x} + v_1 w_{1y} + w_1 w_{1z} + \frac{1}{\rho_1} p_{1z} = 0, \quad (5.4) \]

subject to the boundary conditions

\[ \eta_{1x} + u_1 \eta_{1x} + v_1 \eta_{1y} = \eta_{1}, \quad (5.5) \]

subject to the boundary conditions

\[ \eta_{2x} + u_1 (\eta_{2x} - h_0 x) + v_1 \eta_{2y} = \eta_1, \quad (5.6) \]
\[ \eta_{2x} + u_1 (\eta_{2x} - h_0 x) + v_1 \eta_{2y} = \eta_1, \quad (5.7) \]
\[ u_1 z_1 + r_1 z_1 + v_1 z_1 = 0, \quad (5.8) \]

lower layer:

\[ u_{2x} + v_{2y} + w_{2z} = 0, \quad (5.9) \]
\[ u_{2x} + u_2 u_{2x} + v_2 u_{2y} + w_2 u_{2z} + \frac{1}{\rho_2} p_{2x} = 0, \quad (5.10) \]
\[ v_{2x} + u_2 v_{2x} + v_2 v_{2y} + w_2 v_{2z} + \frac{1}{\rho_2} p_{2y} = 0, \quad (5.11) \]
\[ w_{2x} + u_2 w_{2x} + v_2 w_{2y} + w_2 w_{2z} + \frac{1}{\rho_2} p_{2z} = 0, \quad (5.12) \]

subject to the boundary conditions

\[ \eta_{2x} + u_2 (\eta_{2x} - h_0 x) + v_2 \eta_{2y} = \eta_2, \quad (5.13) \]
\[ p_2 = p_1, \quad (5.14) \]
\[ u_2 z_2 + r_2 z_2 + v_2 z_2 = 0, \quad (5.15) \]

where \( u_i, v_i, w_i \), \( i = 1, 2 \), are the velocities, \( g \) is the gravitational acceleration, \( \rho_i \) are the constant densities, \( p_i \) are the pressures, and \( t \) is the time.

6. Shen's Method

a) Derivation of the Korteweg-de Vries Equation

We define here the large parameter

\[ \sigma^{3/2} = \frac{L}{H} \gg 1, \quad (6.1a) \]

where \( L \) and \( H \) are respectively the longitudinal and transverse length scales. We further introduce the following dimensionless variables:

\[ \left( \frac{g}{H} \right)^{1/2} \sigma^{-3/2} t \rightarrow t, \quad \left( \frac{\sigma^{3/2} x}{H}, \frac{y}{H}, \frac{z}{H} \right) \rightarrow (x, y, z), \]
\[ \Delta = \frac{\rho_1}{\rho_2}, \quad \frac{\eta_1}{H} \rightarrow \eta_i, \quad \frac{D}{H} \rightarrow D, \]
\[ \frac{p_i}{\rho_1 g H} \rightarrow p_i, \quad \frac{h_0}{H} \rightarrow h_0, \quad (6.2a) \]
\[ \left( \frac{u_i}{(g H)^{1/2}}, \frac{\sigma^{1/2} v_i}{(g H)^{1/2}}, \frac{\sigma^{1/2} w_i}{(g H)^{1/2}} \right) \rightarrow (u_i, v_i, w_i). \]
We assume that \( u_i, v_i, w_i, p_i \) and \( \eta_i \), as functions of \( x, y, z \) and \( t \), depend explicitly on the new variable 
\[
\zeta = \sigma \xi (x, t),
\]
where \( \zeta \) will be called a phase function, and that an asymptotic expansion of the form 
\[
\psi_i = \psi_i^{(0)} + \sigma^{-1} \psi_i^{(1)} + \ldots,
\]
where \( \psi_i^{(0)} \) is a vector of the undisturbed uniform state:
\[
(u_i^{(0)}, v_i^{(0)}, w_i^{(0)}) = 0, \quad p_i^{(0)} = -z, \\
p_i^{(0)} = h_0 - \frac{1}{\Delta} (z + h_0), \quad \eta_i^{(0)} = 0.
\]
Substitution of (6.2a)–(6.5a) into (5.1)–(5.15) and use of the expressions 
\[
\psi(z = \eta_1) = \psi(z = 0) + \psi_z \eta_1 + O(\eta_1^2), \\
\psi(z = \eta_2 - h_0) = \psi(z = -h_0) + \psi_z \eta_2 + O(\eta_2^2)
\]
will yield a sequence of equations since the coefficients of like powers of \( \sigma^{-1} \) must be zero. The first-order equations are for the upper layer
\[
k u_1^{(1)} + v_1^{(1)} + w_1^{(1)} = 0, \quad p_1^{(1)} = \Omega u_1^{(1)}, \quad p_1^{(1)} = 0, \quad w_1^{(1)} + \Omega \eta_1^{(1)} = 0, \quad \left\{ \begin{array}{l} p_1^{(1)} = \eta_1^{(1)}, \\ w_1^{(1)} + \Omega \eta_1^{(1)} = 0, \end{array} \right. \quad \text{at } z = 0, \\
v_1^{(1)} D_z + w_1^{(1)} D_z = 0, \quad \text{at } D = 0,
\]
and for the lower layer
\[
k u_2^{(1)} + v_2^{(1)} + w_2^{(1)} = 0, \quad \Delta k p_2^{(1)} = \Omega u_2^{(1)}, \quad p_2^{(1)} = 0, \quad w_2^{(1)} + \Omega \eta_2^{(1)} = 0, \quad \left\{ \begin{array}{l} \eta_2^{(1)} = \frac{\Delta}{1 - \Delta} (p_2^{(1)} - p_1^{(1)}), \\ v_2^{(1)} D_z + w_2^{(1)} D_z = 0, \end{array} \right. \quad \text{at } D = 0,
\]
where
\[
k \equiv \zeta_x, \quad \Omega \equiv -\zeta_t.
\]
From (6.9a) and (6.16a), we may define
\[
p_1^{(1)} \equiv A(x, t, \xi), \quad p_2^{(1)} \equiv B(x, t, \xi).
\]
Multiplying (6.7a) by \( \Omega \) and (6.8a) by \( k \), adding and integrating over the cross section \( S_1 \), we obtain
\[
\frac{k^2 S_1}{\Omega} A_z = \frac{\Delta b \Omega}{\Delta - 1} (B_z - A_z) + a \Omega A_z, \quad \frac{\partial}{\partial z} \left( \frac{k^2 S_1}{\Omega} A_z \right) \equiv \mathcal{X} B_z.
\]
So, we find the linear dispersion relation
\[
a b \Omega^4 - [a S_2(1 - \Delta) + b(\Delta S_2 + S_1)] k^2 \Omega^2 = S_1 S_2(1 - \Delta) k^4 = 0.
\]
Hence, it follows that two modes are possible. From the second-order equations we obtain
\[
p_1^{(2)} = -\varphi_1(x, y, z, t) A_z \xi, \quad p_2^{(2)} = -\varphi_2(x, y, z, t) B_z \xi,
\]
where the potentials \( \varphi_1 \) and \( \varphi_2 \) satisfy the following Neuman problems:
\[
\varphi_{yy} + \varphi_{zz} = k^2, \quad \varphi_{yD} = \varphi_{zD} = 0, \quad \text{at } D = 0, \quad \varphi_{zD} = \Omega^2, \quad \text{at } z = 0, \\
\varphi_A A_z \xi - \varphi_B B_z \xi = \frac{1 - \Delta}{\Delta} \eta_2^{(2)}, \quad \text{at } z = -h_0.
\]
The compatibility condition is reduced to the Korteweg-de Vries equation for the first-order quantity \( B \):
\[
m_1 B_x + m_2 B_x + m_3 B B_z + m_4 B_{zzz} + m_5 B = 0,
\]
where
\[
m_1 = a \Omega \mathcal{X} - \frac{\Delta b (1 - \mathcal{X}) \Omega}{1 - \Delta} + \frac{k^2 S_1 \mathcal{X}}{\Omega}, \quad m_2 = 2k S_1 \mathcal{X}.
\]
K. Murawski • Some Reductive Methods for Nonlinear Waves

\[ m_3 = \frac{\Delta b}{\Delta - 1} \left( 1 - \mathcal{K} \right) k^2 \]

\[ = - \frac{\Delta}{1 - \Delta} \int_{y_{21}}^{y_{22}} \int_{y_{21}}^{y_{22}} \phi_{1} \left| \zeta = -b_0 \right| d \Gamma + \frac{k^4 S_1}{\Omega^2}, \]  

(6.33a)

\[ + \mathcal{K}^2 \int_{y_{21}}^{y_{22}} \int_{y_{21}}^{y_{22}} \phi_{1} \left| \zeta = 0 + k^2 \right| d \Gamma \]

(6.33b)

\[ + \mathcal{K}^2 \mathcal{P} \int_{y_{21}}^{y_{22}} \phi_{1} \left| \zeta = 0 \right| d \Gamma - k^2 \mathcal{K} \int_{y_{21}}^{y_{22}} \phi_{1} d S_1, \]

(6.34a)

Equation (6.24a) may be solved by means of the method of characteristics determined by

\[ \frac{d x}{d t} = \Omega \left( k \right), \quad \frac{d k}{d t} = -k \left( \frac{\Omega}{k} \right), \quad \frac{d \Omega}{d \zeta} = \frac{d \zeta}{d t} = 0. \]

(6.36a)

Along a ray

\[ \frac{d}{d t} = \frac{\partial}{\partial t} + \frac{\partial}{\partial \zeta}, \]

(6.37a)

and (6.30a) may be rewritten as

\[ m_1 B_1 + m_3 B_{\zeta} + m_4 B_{\zeta\zeta} + m_5 B = 0. \]

(6.38a)

A solution of this equation was considered in the context of a solitary wave propagation from one uniform cross section of a symmetric triangular channel into another through a transition region to find the criteria for the fission of solitons. The numerical results showed that the solitary wave is desintegrated into a train of solitons of decreasing amplitudes [17], in qualitative agreement with sea waves. The Korteweg-de Vries equation was also derived by Johnson [34] for a channel of variable depth and appears in various physical systems [35].

b) Derivation of the Modified Korteweg-de Vries Equations

The nonlinear coefficient \( m_3 \) may become zero for some critical value of \( \Delta = \Delta^* \). There a stronger nonlinear effect should be taken into consideration to adopt the expansion

\[ \psi = \psi^{(0)} + \sigma^{-1/2} \psi^{(1)} + \sigma^{-1} \psi^{(2)} + \ldots, \]

(6.1b)

where \( \psi^{(0)} \) is the same undisturbed uniform state vector. The equations for the second approximation \((\sigma^{-1})\) are for the upper layer

\[ k u_1^{(2)} + v_1^{(2)} + w_1^{(2)} = 0, \]

(6.2b)

\[ k p_1^{(2)} + ku_1^{(1)} u_1^{(1)} = \Omega u_1^{(2)}, \]

(6.3b)

\[ p_{1y}^{(2)} = p_{1z}^{(2)} = 0, \]

(6.4b)

\[ k u_1^{(1)} \eta_1^{(1)} = \Omega \eta_1^{(2)} + w_1^{(2)} + w_1^{(1)} \eta_1^{(1)}, \]

(6.5b)

\[ p_1^{(2)} = \eta_1^{(2)} \text{ at } z = 0, \]

(6.6b)

\[ v_1^{(2)} D_x + w_1^{(2)} D_z = 0 \text{ at } D = 0, \]

(6.7b)

\[ k u_1^{(1)} \eta_2^{(1)} + v_1^{(1)} \eta_2^{(1)} \]

\[ = \Omega \eta_2^{(2)} + w_1^{(2)} + w_1^{(1)} \eta_2^{(1)}, \text{ at } z = -h_0, \]

(6.8b)

and for the lower layer

\[ k u_2^{(2)} + v_2^{(2)} + w_2^{(2)} = 0, \]

(6.9b)

\[ k \Delta^* p_2^{(2)} + ku_2^{(1)} u_2^{(1)} = \Omega u_2^{(2)}, \]

(6.10b)

\[ p_{2y}^{(2)} = p_{2z}^{(2)} = 0, \]

(6.11b)

\[ k u_2^{(1)} \eta_2^{(1)} + v_2^{(1)} \eta_2^{(1)} \]

\[ = \Omega \eta_2^{(2)} + w_2^{(2)} + w_2^{(1)} \eta_2^{(1)}, \text{ at } z = -h_0, \]

(6.12b)

\[ \eta_2^{(2)} = \frac{\Delta^*}{\Delta^* - 1} \left( p_2^{(2)} - p_2^{(2)} \right) \text{ at } z = -h_0, \]

(6.13b)

\[ v_2^{(2)} D_x + w_2^{(2)} D_z = 0 \text{ at } D = 0. \]

(6.14b)

Applying the divergence theorem, from \( \sigma^{-3/2} \) we obtain along a ray the modified Korteweg-de Vries equation

\[ n_1 B_1 + n_3 B_{\zeta} + n_4 B_{\zeta\zeta} + n_5 B = 0, \]

(6.15b)

where

\[ n_1 = a \Omega \mathcal{K} - \frac{\Delta^* (1 - \mathcal{K}) b \Omega}{1 - \Delta^*} + \frac{k^2 \mathcal{K} S_1}{\Omega}, \]

(6.16b)
Equation (6.15b) with constant coefficients and

\[ n_5 = 2a k^2 \psi + ak \Omega \tilde{\psi} + \psi \frac{D_2}{D_2 + D_2^{1/2}} \]

\[ \mathcal{J} = C_1 + C_2 \psi, \quad \mathcal{J} = \frac{2 \psi^2 + k^2 \psi^2}{2 \Omega^3}, \]

\[ \psi = \frac{A^* k^2 b \Omega^2 (1 - \psi) + A^* k^4 S_3 (1 - A^*) + 2C_1 \psi^2 [A^*(1 - A^*) k^2 S_2 - A^* \Omega^2 b]}{2 \Omega^2 (C_2 [A^*(1 - 1) k^2 S_2 + A^* b \Omega^2] - A^* b \Omega^2)}, \]

\[ C_1 = \frac{a k^2 \psi^2 (A^* - 1) + k^2 \Omega^2 A^* b \psi (1 - \psi) + k^4 S_1 \psi^2 (A^* - 1)}{2b A^* \Omega^4}, \]

\[ C_2 = \frac{k^2 S_1 (A^* - 1) - A^* b \Omega^2 - a(A^* - 1) \Omega^2}{A^* b \Omega^2}. \]

\[ \varphi_1, \varphi_2 \text{ satisfy (6.26a)-(6.29a) in which } \eta_2(2) \text{ is replaced by } \eta_3(3). \]

7. Summary

In the present paper we reviewed some reductive methods, Taniuti-Wei, reductive expansion, Whitham's, and rays to derive the Korteweg-de Vries, the modified Korteweg-de Vries, the mixed Korteweg-de Vries, and the nonlinear Schrödinger equations. The Korteweg-de Vries equation for liquid waves in a cylinder was previously obtained via Taniuti-Wei [8] and the Whitham's [11] methods and agrees with (3.7), which is found via the derivative expansion method. It supports the convergence of all reductive methods.

It is worth noticing that the two coefficients of the second and third terms in (4.10) are responsible for modulational stability of waves [40]. If their product

\[ A - A^* = \sigma^{-1/2}. \]
is negative, nonlinear wave solutions are modulationally stable. In our case this occurs for arbitrary values of $k$.

The author thanks the referee for valuable comments.

**Appendix**

We rewrite (6.25 b) in the form

$$B_t + x_1 B B_x + x_2 B^2 B_x + x_3 B_{xxx} = 0 \quad (A.1)$$

and look for stationary solutions of the form

$$B = B (\xi = x - c t) \quad (A.2)$$

Upon integration of (A.1) one gets

$$x_3 B_{\xi\xi} - c B + \frac{x_1}{2} B^2 + \frac{x_2}{3} B^3 = a, \quad (A.3)$$

where $a$ is an integration constant. Multiplication of (A.3) with $B_\xi$ yields

$$B_\xi^2 = - \frac{x_2}{6 x_3} B^4 - \frac{x_1}{3 x_3} B^3 + c \frac{x_2}{x_3} B^2 + 2a \frac{x_2}{x_3} B + b,$$

$$b = \text{const.} \quad (A.4)$$

Hence, the phase-diagrams may be sketched (Figs. 2–4).

Fig. 2. Phase-diagrams for $x_2 x_3 < 0$. a: Linear wave limit, b: conoidal wave, c: soliton.

Fig. 3. Phase-diagrams for $x_2 x_3 > 0$. a: Linear wave limit, b: linear wave limit on the left and conoidal wave on the right, c: conoidal waves, d: soliton, e: conoidal wave.

Fig. 4. Phase-diagrams for $x_2 x_3 > 0$ (special case). a: Linear wave limit, b: conoidal wave, c: shock wave.