Chaos-Induced Diffusion.
Analogues to Nonlinear Fokker-Planck Equations
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Deterministic chaotic dynamics induced by a climbing map with variable local jump size is shown to yield anomalous diffusion, i.e. a nonlinear increase of the variance with time. The relation to Fokker-Planck equations with variable drift and diffusivity is given. Turbulent two-particle diffusion may be a possible application.

§1. Introduction
Many investigations [1–7] have recently been devoted to the large-scale diffusive behavior of the solutions of chaotic maps. In [3] we have developed a general approach to the large-scale dynamics of maps as shown in Fig. 1, bottom.

\[ Y_{t+1} = G(Y_t) = Y_t + \tilde{G}(Y_t), \quad -\infty < Y_t < \infty, \quad t = 0, 1, 2, \ldots, \] (1)

where \( \tilde{G}(Y) \) is a 1-periodic function \( \tilde{G}(Y+1) = \tilde{G}(Y) \). The periodicity of \( \tilde{G} \) induces a decomposition of the \( Y \)-space into cells \( [N, N+1) \) of unit length which was used in [3] to derive analytic results for the drift velocity \( \gamma_0 \) and the diffusion constant \( D_0 \) of an ensemble of trajectories of \( \tilde{G} \).

Because of the periodicity of \( \tilde{G} \) the trajectories generated by \( \tilde{G} \) correspond to a chaotic motion in a strictly periodic potential. The phase dynamics of a periodically driven damped anharmonic oscillator (e.g. Josephson junction [8]) may serve as an example.

After a brief review of the results for homogeneous systems in §2 we introduce the conjugating function in §3 which leads to the description of deterministic diffusion in inhomogeneous systems. §4 contains examples of systems with a power-law and an exponential-law inhomogeneity. In §5 we show how the conjugation concept may also be applied to randomly distorted homogeneous systems.

§2. Review of the results for the homogeneous case and generalization to varying cell-sizes
To introduce a generalization of (1) to varying cell-sizes we briefly review some of the results obtained in [3] for the homogeneous case.

The dynamics of (1) can be decomposed into a large-scale part \( N_t \) and a small-scale part \( y_t \). \( N_t \) denotes the integer part of \( Y_t \) and \( y_t \) its fractional part:

\[ Y_t = N_t + y_t, \quad N_t = [Y_t], \quad \text{largest integer } \leq Y_t, \quad y_t \in [0, 1). \] (2)

Then (1) becomes

\[ N_{t+1} = N_t + \Delta(y_t), \quad y_{t+1} = g(y_t), \] (3)

with the jump function

\[ \Delta(y) = [G(y)], \quad y \in [0, 1). \] (4)

and the reduced map

\[ g(y) = G(y) - \Delta(y), \quad y \in [0, 1). \] (5)

The reduced map is assumed to be ergodic and mixing. Here there exists an absolutely continuous invariant density \( \phi(y) \) normalized on \([0, 1)\), which is

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and the diffusion constant
\[ D_0 = \lim_{t \to \infty} \frac{1}{2t} \langle (Y_t - \langle Y_t \rangle)^2 \rangle. \]  
(9)

If the correlation between jumps \( \mathcal{A}(y) \) decays sufficiently fast, \( D_0 > 0 \) is finite \([3, 4]\), and the moments \( \langle Y_t^n \rangle \) behave for large \( t \) like the moments of
\[ P(Y,t) = \frac{1}{\sqrt{4\pi D_0 t}} \exp\left(-\frac{1}{2} \frac{(Y - Y_0 - v_0 t)^2}{2 D_0 t}\right), \]
(10)

which is the solution of the Fokker-Planck equation
\[ \partial_t P(Y,t) = -\partial_y (v_0 - \partial_y D_0) P(Y,t). \]
(11)

This means, one always finds asymptotically normal diffusion with a temporal increase of the variance proportional to \( t \). The details of the reduced map only determine the magnitude of the diffusivity \( D_0 \). Before reaching the asymptotic time regime the variance increases as \( t^2 \) and continuously switches to normal behavior according to the correlation decay of \( \mathcal{A}(y) \) \([5]\). This feature is well known from thermal Brownian motion. The “anomalous” exponent \( \neq 1 \) indicates correlation between successive jumps.

Note that the assumption of the existence of a properly defined invariant density \( \varrho \) excludes an anomalous diffusion caused by an intermittent reduced map. Such a case was described in \([6]\). There the trajectory could get “trapped” by an intermittency center in an interval where \( \mathcal{A}(y) = 0 \). This leads to an algebraically decaying waiting-time distribution and thus to an anomalous diffusion.

§3. Maps with varying cell-sizes

A generalization of (1) to maps with varying cell-sizes can be achieved by conjugation. If \( \{Q_N, Q_{N+1}\} \) is the \( N \)-th cell with a size
\[ l_N = Q_{N+1} - Q_N > Q, \]
(12)

we choose a continuous, piecewise differentiable, invertible function \( h \) with
\[ Q_N = h(N), \quad N = 0, \pm 1, \pm 2, \ldots . \]
(13)

Any \( h \) with these properties will be called a conjugating function. \( X = h(Y) \) maps the dynamical variable \( Y \) onto another dynamical variable \( X \) gov-
erned by the equation
\[ X_{t+1} = F(X_t), \quad t = 0, 1, 2, \ldots \] (14)
with \( F = h \circ G \circ h^{-1} \). \( h \) is a one-to-one mapping of the homogeneous system onto an inhomogeneous system with variable cell-size (Figure 1). The density \( P(Y, t) \) on \( Y \)-space transforms into the \( X \)-space density
\[ R(X, t) = P(h^{-1}(X), t) \left| \frac{dh^{-1}}{dX}(X) \right|. \] (15)
The corresponding Fokker-Planck equation (11) transforms accordingly into
\[ \partial_t R(X, t) = -\partial_X \{ v(X) \partial_X R(X, t) \} \]
with the space-dependent drift and diffusivity
\[ v(X) = v_0 \sqrt{\frac{D(X)}{D_0}} + \frac{1}{2} \frac{dD}{dX}(X), \] (17)
\[ D(X) = D_0 \left[ \frac{dH}{dY} \right]_{Y=h^{-1}(X)}^2. \] (17″)
Thus, the conjugation method yields a class of chaos-induced diffusion processes equivalent to stochastic processes characterized via the general Fokker-Planck equation (16) by a space-dependent drift and diffusivity. If a dynamical system with normal diffusion is used as a starting point, \( v(X) \) and \( D(X) \) are related by (17). There is some analogy to the theory of stochastic processes: a Wiener-process in \( F \)-space drives a more general process in \( X \)-space.

§4. Examples

An example for a deterministic diffusion process with varying drift and diffusivity is turbulent two-particle diffusion [9–11]. If \( X \) is interpreted as the interparticle distance, the diffusivity has to obey Richardson’s 4/3-law [9] in the inertial subrange.
\[ D(X) = D_0 \cdot X^{4/3}. \] (18)
Furthermore, dimensional arguments (Kolmogorov, Oboukhov, v. Weizsäcker, Heisenberg, Onsager; cf. [10, 12]) imply \( r(X) \propto X^{1/3} \). Hence \( r_0 = 0 \) in (17″). Integration of (17″) yields the conjugating function
\[ h(Y) = \frac{1}{27} Y^3. \] (19)
This example motivates the general study of a class of functions \( h_\alpha(Y) \) with power-law behavior. For simplicity we confine ourselves to \( h_\alpha \) with odd symmetry
\[ h_\alpha(-Y) = -h_\alpha(Y). \] (20)
We discuss in particular
\[ h_\alpha(Y) = \begin{cases} Y \cdot Y^{\alpha-1} & \text{if } \alpha > 0, \\ \text{sgn}(Y) \ln(1 + Y^2) & \text{if } \alpha = 0, \\ \text{sgn}(Y) \left[ 1 - (1 + Y^2)^{\frac{\alpha}{2}} \right] & \text{if } \alpha < 0. \end{cases} \] (21)
\( h_\alpha \) generates maps \( F \) where the size \( l_N \) of cells behaves according to
\[ l_N \propto |N|^{\alpha-1} \quad \text{for } N \to \pm \infty. \] (22)
If \( \alpha < 0 \), \( l_N \) decreases so rapidly that the totality of cells only fills a finite interval, which has been normalized to \([-1, 1]\) in (21). Table 1 shows the transport coefficients \( v_\alpha(X) \) and \( D_\alpha(X) \) for the diffusion in \( X \)-space, transformed via \( h_\alpha(Y) \) from the homogeneous Wiener process. The asymptotic \((t \to \infty)\) behavior of the mean \( m_\alpha(t) = \langle X \rangle \), and the variance \( \sigma_\alpha(t) = \langle X^2 \rangle - \langle X \rangle^2 \) can be evaluated in a lengthy and tedious but well-defined way. We only

Table 1. \( X \)-dependence of drift \( v_\alpha(X) \) and diffusivity \( D_\alpha(X) \) for the power-law conjugation (21). For \( \alpha < 0 \) the abbreviation \( Z = 1 - |X| \) is used.

<table>
<thead>
<tr>
<th>( h_\alpha(Y) )</th>
<th>( v_\alpha(X) )</th>
<th>( D_\alpha(X) )</th>
</tr>
</thead>
<tbody>
<tr>
<td>( Y \cdot Y^{\alpha-1} ), ( \alpha &gt; 0 )</td>
<td>( x \cdot r_0 + (x - 1) D_0 \text{sgn}(X) \cdot [X^{\alpha-1/2} \cdot</td>
<td>X</td>
</tr>
<tr>
<td>( \text{sgn}(Y) \ln(1 + Y^2) ), ( \alpha = 0 )</td>
<td>( 2 \left{ [r_0 - D_0 \text{sgn}(X) \cdot e^{-X^2/2} - 2e^{-3/2 - X^2/2}] / \left[ 1 - e^{-X^2} \right]^{1/2} \right} \cdot [e^{-X^2} - e^{-2X}]^{1/2} )</td>
<td>( 4 D_0 \cdot [e^{-X^2} - e^{-2X}] )</td>
</tr>
<tr>
<td>( \text{sgn}(Y) \cdot [1 - (1 + Y^2)^{\frac{\alpha}{2}}] ), ( \alpha &lt; 0 )</td>
<td>( z \left{ [r_0 - (1-z) D_0 \text{sgn}(X) \cdot Z^{-\alpha/2} - 2z] / \left[ 1 - Z^{-2\alpha/3} \right]^{1/2} \right} )</td>
<td>( x^2 D_0 \cdot [Z^{-2/3} - Z^{-4/3}] )</td>
</tr>
</tbody>
</table>
Fig. 2. Variances $\sigma_x$ vs. time $t$ for various $x$. The data points represent the numerically computed $\sigma_x$ of an ensemble of 5000 points which were started in the first cell at $t = 0$. The lines indicate the behavior to be expected from the asymptotic formulae in Table 2. Note the slow convergence of data points to the asymptotic behavior near the localization transition $x = 0$. The results for $a = 0.5$ have been multiplied by a factor of 10 for clarity of representation.

Fig. 3. The plot shows the exponent of $t$ in the leading term as a function of $x$ for power-law conjugation. The dotted lines refer to $m_x$, the dashed lines to $\sigma_x$. The symbols indicate additional factors of $\ln t$ in $m_x$ (●), $\sigma_x$ (■) as well as $(\ln t)^{\alpha}$ in $\sigma_x$ (□).

report the results in Table 2 and compare with numerical data in Figure 2.

If $x = 1$ we recover normal diffusion. Other positive $x$ yield anomalous diffusion. For negative $x$ the dominating feature is the localization.

One important aspect of the results presented in Table 2 is the behavior of the exponents of $t$ in the leading-order $t$-dependent terms of $m_x$ and $\sigma_x$ when $x$ is changed. As one can see (Fig. 3), exponents are bounded below by $-1/2$ if $t_0 = 0$. The transition from the finite system ($x < 0$) to the infinite system ($x > 0$) appears as a critical situation where $m_x$ as well as $\sigma_x$ contain logarithmic corrections. $\sigma_x$ undergoes another transition at $x = -1$, which indicates the following qualitative change in the dynamics: For $x > -1$ the increase of $\sigma_x$ with time is still a consequence of a spreading over more and more cells. For $x < -1$ the size of cells decreases so rapidly that the ensemble is spread over essentially all of the interval $[-1, 1]$ after a few time-steps. The further change of $\sigma_x$ is then brought about by a decrease of the probability in the core of $[-1, 1]$ due to the normalization prefactor $\propto t^{-1/2}$ of $P(Y,t)$. If $t_0 = 0$, exponents are not bounded below. The drift "squeezes" the diffusing ensemble against the boundary of the system. The smaller $x < 0$ becomes, the faster do the cells decrease in size when the boundary is approached. Therefore, the width $\sigma_x$ of the ensemble shrinks and $m_x$ approaches $± 1$ the faster, the smaller $x$ is. Again $m_x$ has a logarithmic correction for $x = 0$. However, $\sigma_x$ does not. This can be explained as well by the squeezing effect. While the mean-motion really undergoes a qualitative change when the finite system is replaced by the infinite one (there are no longer any boundaries to stop it), the width does not experience the limitation due to finiteness of space since it is shrinking anyhow, because the ensemble is squeezed into smaller and smaller cells. This is already present for positive $x < 1/2$.

Though for large $x > 0$ $\sigma_x$ increases with an arbitrarily high power of $t$, $D(X) \propto X^{2 - 2/\alpha}$ always increases more slowly than $X^2$ for $X \to \infty$. To have $D(X) \propto X^2$ requires an exponentially changing $h_x$. So we consider as a second example

$$h_x(Y) = \begin{cases} \sinh(\alpha Y), & x > 0, \\ \text{sgn}(Y) \left|1 - e^{2 Y}\right|^{-1/2}, & x < 0. \end{cases} \quad (23)$$

The results for $v_x$, $D_x$, $m_x$, and $\sigma_x$ are given in Tables 3 and 4, respectively. Because of the symmetry (20)
Table 2. Asymptotic time-dependence of the mean \( m_x(t) \) and the variance \( \sigma_x(t) \) for the power-law conjugation (21).

\[ v_0 = 0: \]

<table>
<thead>
<tr>
<th>( x )</th>
<th>( m_x(t), \ t \rightarrow \infty )</th>
<th>( \sigma_x(t), \ t \rightarrow \infty )</th>
</tr>
</thead>
<tbody>
<tr>
<td>( x &gt; 0 )</td>
<td>( \Gamma \left( 1 + \frac{x}{2} \right) \frac{2}{\sqrt{\pi}} Y_0 (4 D_0 t)^{(x-1)/2} )</td>
<td>( \Gamma \left( 1 + \frac{x}{2} \right) \frac{1}{\sqrt{\pi}} (4 D_0 t)^2 )</td>
</tr>
<tr>
<td>( x = 0 )</td>
<td>( \frac{2}{\sqrt{\pi}} Y_0 (\ln (4 D_0 t) - \gamma) (4 D_0 t)^{-1/2} )</td>
<td>( [\ln (4 D_0 t)]^2 )</td>
</tr>
<tr>
<td>( -1 &lt; x &lt; 0 )</td>
<td>( \frac{2}{\sqrt{\pi}} Y_0 (4 D_0 t)^{-1/2} )</td>
<td>( 1 - \Gamma \left( 1 + \frac{x}{2} \right) \frac{2}{\sqrt{\pi}} (4 D_0 t)^{2x/2} )</td>
</tr>
<tr>
<td>( x = -1 )</td>
<td>( -2 \ln (t_0) )</td>
<td>( 2 - (v_0 t)^{-2} (4 D_0 t) )</td>
</tr>
<tr>
<td>( x &lt; 1 )</td>
<td>( 1 - (v_0 t)^x )</td>
<td>( \frac{2}{2} (v_0 t)^{2(1-x)} (4 D_0 t) )</td>
</tr>
</tbody>
</table>

\( \gamma = 0.577215664 \ldots \) Euler’s constant

\[ v_0 > 0: \]

<table>
<thead>
<tr>
<th>( x )</th>
<th>( m_x(t), \ t \rightarrow \infty )</th>
<th>( \sigma_x(t), \ t \rightarrow \infty )</th>
</tr>
</thead>
<tbody>
<tr>
<td>( x &gt; 0 )</td>
<td>( (v_0 t)^x )</td>
<td>( \frac{x^2}{2} (v_0 t)^{2(x-1)} (4 D_0 t) )</td>
</tr>
<tr>
<td>( x = 0 )</td>
<td>( 2 \ln (t_0) )</td>
<td>( 2 - (v_0 t)^{-2} (4 D_0 t) )</td>
</tr>
<tr>
<td>( x &lt; 0 )</td>
<td>( 1 - (v_0 t)^x )</td>
<td>( \frac{x^2}{2} (v_0 t)^{2(1-x)} (4 D_0 t) )</td>
</tr>
</tbody>
</table>

Table 3. \( X \)-dependence of drift and diffusivity for the exponential conjugation (23).

<table>
<thead>
<tr>
<th>( h_x(Y) )</th>
<th>( v_x(X) )</th>
<th>( D_x(X) )</th>
</tr>
</thead>
<tbody>
<tr>
<td>( \sinh (x Y), \ x &gt; 0 )</td>
<td>( x \left[ t_0 + x D_0 \frac{X}{\sqrt{1+X^2}} \right] \frac{1}{\sqrt{1+X^2}} )</td>
<td>( x^2 D_0 \left[ 1 + X^2 \right] )</td>
</tr>
<tr>
<td>( \text{sgn} (Y) [1 - e^{2 Y}], \ x &lt; 0 )</td>
<td>( x \left[ t_0 - x D_0 \text{sgn}(X) \right] \frac{1}{1 -</td>
<td>X</td>
</tr>
</tbody>
</table>

The mean \( m_x \) and the variance \( \sigma_x \) have the symmetry

\[ m_x(Y_0, v_0) = - m_x(-Y_0, -v_0), \]
\[ \sigma_x(Y_0, v_0) = \sigma_x(-Y_0, -v_0). \]  

Therefore the results for \( t_0 \geq 0 \) given in Tabs. 2 and 4 can easily be converted to \( t_0 \leq 0 \).

The dynamical system defined by \( h_x \) in the \( X \)-space has two typical “length” scales, an inner scale \( l_q \), which is the size of the smallest (largest) cell if \( l_N \rightarrow \infty \) \((l_N \rightarrow 0)\) for \( N \rightarrow \infty \) and the total range of diffusion \( L \), which is finite for \( x < 0 \). Introducing parameters \( p, q > 0 \) one can adjust the inner and outer scales to their proper values by using

\[ \tilde{h}_x(Y) = p h_x(q Y). \]  

The corresponding moments \( \tilde{m}_x \) and \( \tilde{\sigma}_x \) are related to those given in the tables by

\[ \tilde{m}_x(t; Y_0, v_0, D_0) = p m_x(t; q Y_0, q v_0, q^2 D_0), \]
\[ \tilde{\sigma}_x(t; Y_0, v_0, D_0) = p^2 \sigma_x(t; q Y_0, q v_0, q^2 D_0). \]
Table 4. Time-dependence of the mean \( m(x) \) and the variance \( \sigma(x) \) for the exponential conjugation (23). For \( x > 0 \) the complete expressions are given, for \( x < 0 \) the asymptotically leading terms, only.

\[
\begin{align*}
\text{for } x > 0: & \quad m(x) = \sinh \left[ x \left( Y_0 + v_0 t \right) \right] e^{2x D_0 t} \quad \sigma(x) = \frac{1}{2} \left[ e^{2x D_0 t} - 1 \right] \\
\text{for } x < 0: & \quad \lim_{t \to \infty} m(x) = \frac{3}{2} \left( \frac{2x}{x} \right) \left( \frac{D_0}{v_0} \right)^{-1/2} \\
& \quad \lim_{t \to \infty} \sigma(x) = \frac{1}{2} \left( \frac{2x}{x} \right) \left( \frac{D_0}{v_0} \right)^{-1/2}
\end{align*}
\]

The attentive reader will have noticed that we have specified the conjugating function in §3 on the cell boundaries only. In between, \( h_x \) is arbitrary up to the restrictions of continuity and monotonicity. In fact, the details of \( h_x \) determine the fine structure of the probability density within the cells. But they do not have an influence upon the integral weight of each cell. The results obtained with the particular \( h_x \) specified in (21) and (23) thus also hold for \( h_x \) with different fine-structure if changes inside cells are negligible with respect to the overall features, i.e. \( l_0 \ll L \).

§5. Random spatial fluctuations

The conjugation method can also be applied when the cell-sizes \( l_0, N = 0, \pm 1, \pm 2, \ldots \) are random variables. To demonstrate this we consider a particular example:

We assume the \( l_0, N = 0, \pm 1, \pm 2, \ldots \) to be identically distributed variables with a mean \( \bar{l} \). The fluctuations \( \delta l_N = l_N - \bar{l} \) are supposed to be mutually independent, i.e. \( \delta l_N \delta l_{N'} = (\delta l)^2 \delta_{NN'} \) with a finite variance \( (\delta l)^2 < \infty \). The overbar denotes averaging with respect to the cell-size statistics. Introducing the notation

\[
\sum_{n=0}^{N} \begin{cases} 
N & \text{if } N > 0, \\
0 & \text{if } N = 0, \\
-1 & \text{if } N < 0,
\end{cases}
\]

we define a stochastic conjugating function by

\[
h(N) = \sum_{n=0}^{N} l_n = N \bar{l} + \sum_{n} \delta l_n .
\]

To calculate \( m(t) \) and \( \sigma(t) \) we use a decomposition of \( X = h(Y) \) based on the \( Y \)-decomposition (2)

\[
X_i = h(N_i) + l_i x_i \quad \text{with } \quad x_i \in [0,1].
\]
Thus, one finds up to order $O(t^0)$ for the mean
\[ m(t) = \langle X \rangle_t, \]
\[ m(t) = \bar{I}\langle N \rangle = \bar{I}t_0 t. \]  
(30)
The leading order for the variance \( \sigma(t) = \langle X^2 \rangle_t - \langle X \rangle_t^2 \) reads
\[ \sigma(t) = \bar{I}^2 (\langle N^2 \rangle_t - \langle N \rangle_t^2) + (\bar{I}t_0)^2 \langle N \rangle_t + O(t^0) \]
\[ = \begin{cases} 
(2\bar{I}^2 D_0 + (\bar{I}t_0)^2) t + O(t^0) & \text{if } t_0 \neq 0, \\
2\bar{I}^2 D_0 t + (\bar{I}t_0)^2 \sqrt{\frac{4D_0}{\pi}} t^{1/2} + O(t^0) & \text{if } t_0 = 0,
\end{cases} \]  
(31)
Note, the disorder in cell-sizes does not cause anomalous diffusion for \( t \to \infty \) if, as assumed here, the cell-size fluctuations have a finite variance. The anomalous $t^{1/2}$-term dominates for small \( t \) only, i.e. if \( \sqrt{\pi D_0 t} < (\bar{I}t_0)^2 / \bar{I}^2 \). If, however, the cell-size variance is infinite, one finds truly anomalous diffusion, as a proper discussion of (31) proves.

Since any such disordered system is topologically conjugate to the homogeneous \( Y \)-system by (28), the disorder is in the cell-sizes only. The cell-to-cell transition rates do not fluctuate. Therefore, these systems with random disorder correspond rather to stochastic systems with anomalous diffusion as described in [13] (fluctuations of lattice constant) then to those with randomly disordered transition rates (see e.g. [14]).

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