Analytical Evaluation of the Hadron Number in Extensive Air Showers for Rising Inelasticity

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A purely analytical method is presented in this paper for evaluating the average number of hadrons (in an air shower) with a given energy as the inelasticity of the cross-section rises. Our result is found to be in excellent agreement with numerical solutions and also with experimental data, particularly in the high energy range.

I. Introduction

In view of its importance for multiparticle production processes in the ultra high energy region the analysis of the hadronic components of extensive air showers (EAS) has become an active field of interest. A crucial point of such an analysis consists in determining the average number of hadrons with a given energy. Unfortunately, most of the studies employ numerical techniques involving a clattering mass of experimental data and a lot of equipment.

The present paper employs a purely analytical method to determine the average number of hadrons with a given initial energy as the inelasticity of the cross-section rises. In a recent paper [1] a solution for nucleons has been found; in the pion case the diffusion equation itself is somewhat harder but our previous approach works, mutatis mutandis, quite satisfactorily. As in the previous paper [1] we shall follow the rigorous saddle-point method which we developed in a recent work [2] for the inversion of the Mellin transform.

The raison d’être of the present paper will be apparent when one considers the following facts:

In the first place the coupled integro-differential equations that occur in the present problem have prompted one numerical solution after another [3]–[5] although they are by no means intractable analytically. Perusing some of them which are found to be protracted over sixteen odd years, we have undertaken here to vindicate the feasibility of a purely analytical solution. For the general reader in particular an analytical solution is always more suggestive of novel ideas than any cut and dried numerical sheet. Secondly, the prevalent numerical solutions employ “stronger” assumptions than we need even in our wholly analytical approach. In a standard numerical calculation [3] the nucleon interaction length is approximated by a constant (see the paragraph preceding Eq. (2.4) of [3]). In another [4] $\lambda_\pi$ and $\lambda_N$ are accorded identical values throughout. We, however, consider a more general situation in the sense that we do not require $\lambda_\pi$ and $\lambda_N$ to be equal. Of course, the present work too needs some assumption. We assume $\alpha_\pi = \alpha_N$ (see Eqs. (4)–(6) below); but then, this is evidently a more tenable hypothesis than $\alpha_\pi = \alpha_N$.

II. Analytical Evaluation of the Hadron Number

Consider the relevant integro-differential equations (of diffusion):

$$\frac{\partial N}{\partial \rho} = -\frac{N}{\lambda_N} + \int E \frac{D_N(E', P)}{E} W_{NN}(E', E) dE', \quad (1)$$

$$\frac{\partial \Pi}{\partial \rho} = -\frac{1}{\lambda_\pi} \Pi + \frac{B}{E} \Pi + \int E \frac{D_\pi(E', P)}{E} W_{NP}(E', E) dE'$$

$$+ \int E \frac{\Pi(E', P)}{E} W_{\pi\pi}(E', E) dE', \quad (2)$$

where $\lambda_{m, \pi}$ etc. are written $\lambda_\pi$ etc. for simplicity, the rest of the symbols being as in [5] and [6].

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Assuming $B/E_p \ll 1$ and setting $II = F_\pi e^{-p/\bar{\lambda}_\pi}$ and $p' = p/\bar{\lambda}_\pi(E)$, we reduce (2) into:

$$\frac{\partial F_\pi}{\partial p'} = \int \frac{E_0}{E} \frac{\lambda_\pi(E)}{\lambda_\pi(E')} \exp\left\{ p/\bar{\lambda}_\pi(E) \right\} F_N$$

$$\cdot \exp\left\{ -p/\bar{\lambda}_\pi(E') \right\} W_{\pi N}(E', E) \, dE'$$

$$+ \int \frac{E_0}{E} \frac{\lambda_\pi(E)}{\bar{\lambda}_\pi(E')} \exp\left\{ \frac{p}{\bar{\lambda}_\pi(E)} - \frac{1}{\bar{\lambda}_\pi(E)} \right\}$$

$$\cdot F_N \, W_{\pi N}(E', E) \, dE', \quad (3)$$

where we have written

$$N(E', p) = F_N(E', p) \exp\left\{ -p/\bar{\lambda}_\pi(E') \right\}.$$

We next express the rise of inelasticity in the standard form:

$$\sigma_{\text{in}} = \sigma_0 \left( 1 + a \ln N \right),$$

i.e.

$$\frac{1}{\bar{\lambda}_\pi N} = \left( 1 + a_{\pi N} \ln \frac{E}{100} \right) \frac{1}{\bar{\lambda}_N^0} \quad \text{(4)}.$$

As noted before, we do not require $\bar{\lambda}_\pi = \bar{\lambda}_N$ but use the somewhat more generally valid assumption $\bar{\lambda}_\pi = \bar{\lambda}_N = a$ (say).

From (4), writing $X = E/E'$, we obtain

$$\frac{\lambda_\pi(E)}{\lambda_\pi(E')} \approx 1 - a \ln X, \quad \text{(5)}$$

and

$$\frac{\lambda_\pi(E)}{\lambda_N(E')} \approx \frac{\lambda_0^0}{\lambda_N^0} (1 - a \ln X). \quad \text{(6)}$$

Also

$$\frac{1}{\lambda_\pi(E)} - \frac{1}{\lambda_N(E')} \approx \frac{1}{\lambda_0^0} - \frac{1}{\lambda_N^0} + \frac{a}{\lambda_N^0} a \ln X, \quad \text{(7)}$$

the last equation following from a neglect of second order of smallness, viz. $a \left( \frac{1}{\lambda_0^0} - \frac{1}{\lambda_N^0} \right)$.

Accordingly (3) reduces to ($w = E/E_0$)

$$\frac{\partial F_\pi}{\partial p'} (E, p') = \frac{\lambda_0^0}{\lambda_N^0} \int_{w}^{1} f_{\pi N}(X) \frac{dx}{X^2} \left( 1 - a \ln X \right) F_N$$

$$\cdot \exp\left\{ a p' \ln X \right\} \cdot \exp\left\{ p' \left( 1 - \frac{\lambda_0^0}{\lambda_N^0} \right) \right\}$$

$$+ \int_{w}^{1} f_{\pi N}(X) \frac{dx}{X^2} \left( 1 - a \ln X \right) F_\pi$$

$$\cdot \exp\left\{ a p' \ln X \right\}, \quad (8)$$

where we have introduced a naive form of Feynman scaling typified by

$$f_{\pi N}(X) = E W_{\pi N}(E, E') = \frac{1}{\sigma_{\text{in}}} \int E \frac{1}{x^3} \frac{dE'}{p^3} \, dp'.$$

Subjecting (8) to Mellin transform (defined by $M_\pi = \int_{0}^{\infty} F_\pi(E^s) \, dE$; $M_N = \int_{0}^{\infty} F_N(E^s) \, dE$), we have for large $E_0$,

$$\frac{\partial M_\pi}{\partial p'} = g(s) e^{p' g(s)} M_N(s, p') + M_\pi(s, p') g(s) \quad \text{(9)}$$

where

$$g(s) = \int_{0}^{1} x^{s+a p' -1} f(X) \left( 1 - a \ln X \right) \, dx$$

and

$$\epsilon = 1 - \frac{\lambda_0^0}{\lambda_N^0}.$$

Equation (9) may be considered a linear differential equation for $M_\pi$ whose solution (ignoring the $p'$ dependence of $g(s)$ which is inconsequential in view of the smallness of $a$) is given by

$$M_\pi = C e^{p' \bar{g}(s)} + e^{p' \bar{g}(s)} \frac{\lambda_0^0}{\lambda_N^0} \int_{s}^{1} g(s) e^{(e-\bar{g}(s)) p'} M_N(s, p') \, dp', \quad (10)$$

where $C$ is an arbitrary integration constant.

But

$$M_N(s, p') = \int_{0}^{\infty} F_N(E^s) \, dE \approx M_N(s, 0) e^{p' \bar{g}(s)},$$

where $\bar{g}(s)$ is related to the process $NN \rightarrow p + X$. Since $M_N(s, 0) = 0$, $C$ in (10) can be known.

Also, for $\bar{g}(s) + \epsilon < g(s)$, we have

$$M_\pi \approx M_N(s, 0) [e^{p' \bar{g}(s)} - e^{p' \bar{g}(s)}] \quad \text{(11)}.$$

Again,

$$M_N \approx M_N(s, 0) e^{p' \bar{g}(s)}. \quad \text{(12)}$$

Hence

$$M_\pi + \frac{\lambda_0^0}{\lambda_N^0} M_N = \frac{\lambda_0^0}{\lambda_N^0} M_N(s, 0) e^{p' \bar{g}(s)}. \quad \text{(13)}$$

What is still left is just the inversion of this Mellin-transformed relation. This is found from our previous work of inversion [2] by a comparison. We
accordingly find
\[
\Pi + \frac{C_0}{E} \frac{\pi}{p'} N = \frac{C_0}{E} \frac{\pi}{p'} \sqrt{\frac{\pi}{p'}} \exp \left[ p' \left( g(s_0) - 1 \right) - s_0 \ln W \right] .
\]  

(14)

Also, from another previous work of ours [1],
\[
N = \frac{C_0}{E} \sqrt{\frac{\pi}{p''}} \exp \left[ p' \left( g(s_0) - 1 \right) - s_0 \ln W \right] ,
\]  

(15)

where \( p'' = \frac{p}{\pi_N(E)} \) and \( s_0 \) is the saddle-point obtained by solving (1). Note that the elegance of the solution (14) is that the only free parameter here is \( C_0 \); for details of the inversion procedure we allude to our previous works [1, 2]. From (14) and (15), then, the total hadron spectrum \( \Pi + N \) can easily be obtained. If, however, we use \( \frac{\pi_0}{\pi_N} \approx \frac{\pi_0}{\sqrt{E}} \), (13) alone would yield the total hadron spectrum:
\[
\Pi + N = \frac{C_0}{E} \sqrt{\frac{\pi}{p'}} \exp \left[ p' \left( g(s_0) - 1 \right) - s_0 \ln W \right] ,
\]  

(16)

where \( s_0 \) is given by
\[
\frac{dq(s)}{ds} \bigg|_{s=s_0} = \frac{\ln W}{p'}
\]  

and \( s_0 = s_0 - a p' + 1 \).

Note that in particular, for the process \( NN \rightarrow \Pi^\pm + X \), \( f(X) \) can be simulated by the formula [7]
\[
f(X) \sim A_0(a - X)^a.
\]

Then
\[
g(s) = \int_0^{1} X e^{-a p - 1} f(X) (1 - a \ln X) dX
\]  

\[\cong A_0 B (s + a p', s), \]

(17)

where \( B \) is the usual beta function. In that event \( s_0 \) admits of an approximate representation in terms of

\[
\frac{d}{ds} B(s + a p', s) \bigg|_{s=s_0} = \frac{\ln W}{A_0 p'} .
\]  

(18)

The last expression can be solved by successive approximation.

For comparing the result of our calculation with a standard work [8] we use \( a = 0.03 \) and obtain excellent agreement as evinced by Fig. 1, which moreover compares our result with the numerical calculations [5] especially in the higher reaches of the energy range.

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