1. Introduction

In stellarator configurations the displacement of the magnetic axis due to the pressure effect is in leading order linearly proportional to $\beta$ \cite{1,2} ($\beta = p/(p + B^2/2) = $ kinetic pressure over the total pressure at the magnetic axis). Limiting the displacement to a fixed fraction of the plasma radius defines a maximum value $\beta_e$ of beta. This we take as an estimate of the maximum accessible equilibrium value $\beta_e$ of beta. In the case of an $l=2$ stellarator with a circular magnetic axis the asymptotic theory \cite{1} gives* $\beta_e = 2\zeta_0/\zeta$; here $A = R_2/b$ is the aspect ratio of the torus related to the mean radius $b$ of the cross-section of the outer conducting wall; $\zeta_0$ is the twist number along the full torus at the magnetic axis (angle of rotational transform divided by $2\pi$). This estimate for $\beta_e$ was derived under the conditions $1/A < 1$, $M/A = 2\pi b/L_\parallel < 1$; $M$ is the number of field periods around the full torus (so the number $N$ of wire periods is given by $N = M/2$ for an $l=2$ configuration). According to Shafranov \cite{1} the magnetic axis is displaced by an amount of $b/2$ by considering only terms linear in beta if $\beta$ approaches the value $\beta_e$; considering also the nonlinear terms, the displacement is asymptotically limited to $b/\sqrt{3}$ for large values of $\beta/\beta_e$.

Since the shear of an $l=2$ vacuum field is of the order $M/2A$, it can certainly not be neglected when $M \approx A$. Danilkin \cite{3} found for large-shear stellarator fields that the above estimate of the equilibrium value of $\beta$ is about one order of magnitude too small. In the case of low-\$\beta$, small-shear $l=2$ configurations the maximum accessible beta value obtained by Danilkin is of the order of that given by Shafranov \cite{1}, Greene et al. \cite{2}, and that reported here.

Greene et al. \cite{2} showed for $l=2$ stellarators which have a magnetic hill rather than a magnetic well that the outward shift of the plasma column becomes singular at a certain value of $\beta$. At this value of $\beta$ the plasma is marginally stable with respect to an $m=1, k=0$ mode (rigid shift). Further estimates of the critical $\beta_c$ for equilibrium and stability in $l=1, l=2, l=3, \ldots$ configurations were given by Grad et al. \cite{4} by using various expansion schemes (low $\beta$ and finite $\beta$) and by considering the stability of certain global modes. Mikhailov \cite{5} re-analyzed this problem and discussed the effect of equilibrium beta and magnetic hill. He confirmed the results of \cite{2}.

The aim of the present paper is to study the above effects with the help of three-dimensional finite-beta equilibrium solutions of the magnetohydrostatic boundary value problem in a toroidal.
domain. While analytical methods (e.g. [1, 6]) and several fully three-dimensional codes (e.g. [7, 8, 9]) (with no other approximations than that of the finite grid size) are available, the code by Bauer, Betancourt, and Garabedian [7] (denoted by BBG code in the following) turned out to be an especially versatile tool and is used here. In Sect. 2 the coordinate system, the boundary conditions, and the choice of the invariant functions (rotational transform and mass distribution as functions of the magnetic surfaces) are described. Numerical convergence studies (to zero mesh size) for the energy, and both the average displacement and the helical \( l = 1 \) deformation of the magnetic axis in \( l = 2, l = 1, 2 \) stellarators, and \( l = 0, 1, 2, 3 \) configurations are performed in Sects. 3 and 4. The dependence of the axis displacement and the helical deformation of the magnetic axis on \( \beta \) is given in Sect. 3 for \( l = 2 \) and in Sect. 4 for more general \( l = 0, 1, 2, 3 \) stellarator configurations, and is compared with analytical results.

The equilibrium computations are performed over one field period of length \( L_P \). Detailed stability investigations of these equilibria are beyond the scope of this paper. However possible instabilities, saturation phenomena, and interaction of unstable modes obeying the same toroidal periodicity conditions (in particular the rigid radial disturbance corresponding to an \( m = 1, k = 0 \) mode) are inherently studied as well. The parameter studies (beta variation) of the equilibrium computations are performed in such a way that the rotational transform (twist number \( i \)) as function of the labeling coordinate \( s \) is prescribed in a given configuration and the \( \beta \)-value is changed from case to case by changing an \( s \)-independent factor in the mass function.

The iteration procedure to minimize the potential energy is continued until typically about seven decimal figures of the potential energy remain constant.

A comparison of results on high-\( \beta \), \( l = 0, 1, 2 \) stellarator equilibrium computations, including a vacuum field outside the plasma, with the ISAR T1-B experiment [10] was made by Herrnegger et al. [11]. A synopsis of results with the BBG code (further finite-beta results in advanced stellarators with reduced Pfirsch-Schlüter currents are reported in [12, 13]) and the Chodura-Schlüter code [8] is given in [14].

2. Boundary Conditions and Choice of Invariant Functions

2.1. Coordinate System

The coordinate system is shown in Figure 1. The transformations from Cartesian coordinates \((x, y, z)\) to orthogonal modified cylindrical coordinates \((r, \theta, z)\) having the origin at \( R_T \) are defined by

\[
\begin{align*}
x &= (R_T + r) \cos \theta, \\
y &= (R_T + r) \sin \theta, \\
z &= z.
\end{align*}
\tag{1}
\]

The plasma is surrounded by an ideally electrically conducting wall which is a magnetic surface. A new coordinate system \((s, u, v)\) is introduced in the plasma region tied to the equilibrium magnetic surfaces, the origin of which is at the magnetic axis. A set of nested toroidal flux surfaces is assumed each of which is used as Lagrangian coordinate surface and labelled by \( s \); \( s \) has the value \( s = 0 \) at the magnetic axis and the value \( s = 1 \) at the outer conducting wall. The coordinate \( v \) varies in the range \( 0 \leq v \leq 1 \) and is related to the toroidal angle \( \theta \) by \( \theta = 2\pi v / M \), where \( M \) is the number of field periods around the torus.

The full torus consists of \( M \) equal sectors and is passed once for \( 0 \leq \theta \leq 2\pi \). The length \( L_P \) of a field period is measured along the circle with the torus radius \( R_T \) and is given by \( L_P = 2\pi R_T / M \). The relation between the toroidal aspect ratio \( A \), the helical aspect ratio \( A_h \), and \( M \) is given by \( A_h = L_P / 2\pi b = A / M (\epsilon = 2\pi b / L_P \) is the periodicity number of the field period). The coordinate \( u \) \((0 \leq u \leq 1)\) parameterizes the poloidal angle in meridional planes.

The magnetic axis has the parameter representation \( r_a(v), z_a(v) \). The dimensionless radius function \( R(s, u, v) \) is related to \( r \) and \( z \) by the equations

\[
\begin{align*}
r &= r_a(v) + R(s, u, v) [r_a(u, v) - r_a(v)], \\
z &= z_a(v) + R(s, u, v) [z_a(u, v) - z_a(v)].
\end{align*}
\tag{2}
\]

\[\text{Fig. 1. Coordinate system.}\]
where \( r_b(u, v), z_b(u, v) \) is the parameter representation of the outer conducting wall. As an initial condition, \( R \) is taken to be a function of \( s \) only: \( R^2 = s(1 + c s)/(1 + c) \); here \( c \) can be chosen arbitrarily; typical values are \( c = 0, 2, 4 \) for the following study.

The shape of the outer conducting wall was in the earlier part of this work represented by

\[
\begin{align*}
r_b(u, v)/b &= g_b \cos U + A_1(1 - \alpha) \cos V - A_2 \cos (U - V), \\
z_b(u, v)/b &= g_b \sin U + A_1(1 + \alpha) \sin V + A_2 \sin (U - V),
\end{align*}
\]

where \( U = 2\pi u, V = 2\pi v \),

\[
g_b(u, v) = 1 - A_0 \cos V + \sum_{l=1, 2, \ldots} A_{10} \cos lU - \sum_{l=3, 4, \ldots} A_l \cos (lU - V) + \sum_{l=2, 3, \ldots} \sum_{m=2, 3, \ldots} A_{lm} \cos (lU - m V).
\]

A modified boundary representation avoiding the formation of trefoils at larger amplitudes of the corrugations has been found to be more advantageous [13] and is given in the figure caption of Figure 17.

The helical \( l = 0, 1, 2, \ldots \) corrugations corresponding to the dimensionless amplitudes \( A_l \) are periodic in \( v \) with the period unity (corresponding to the period length \( L_p \)). In particular, the \( A_2 \)-deformation of the boundary is an exact ellipse in the planes \( v = \) constant; they revolve uniformly around their geometric centre if \( v \) is increased from zero to one. The relation between their half-axis ratio \( e \) and the ellipticity \( A_2 \), is given by \( A_2 = (1 - e)/(1 + e) \).

The period of the helical corrugations corresponding to \( A_{lm} \) is given by \( 1/m \) (corresponding to the length \( L_p/m \)); \( A_3 \) and \( A_{32} \) describe helical \( l = 3 \) corrugations which revolve by an angle of \( 2\pi/3 \) and \( 4\pi/3 \), respectively, per field period length \( L_p \). Axisymmetric corrugations are denoted by \( A_{10} \) (\( l = 1, 2, \ldots \)).

The geometric centre of the cross-sections moves on a toroidal helix-like curve wound on a torus of elliptical cross-section with half-axes \((1 - \alpha)A_1 \) and \((1 + \alpha)A_1 \). The deviation from a circular torus is described by \( \alpha \); at \( \alpha = -1 \) the helicoidal line degenerates to a plane, non-circular toroidal curve oscillating around the circle of given radius \( R_T + A_{10} \).

The sign of the corrugation \( A_1 \) is chosen such that at \( v = 0 \) (standing ellipse for \( A_2 > 0 \)) the helical curve of the geometric centre of the boundary contours starts at the outer edge of the torus for \( A_1 > 0 \) and at the inner edge for \( A_1 < 0 \).

The “bumpiness” of the configuration with period length \( L_p \) is denoted by \( A_0 \); it describes the ratio of the cross-sectional area at \( v = 0 \) and \( v = 1/2 \); \( A_0 > 0 \) means that the area at \( v = 0 \) is smaller than that at \( v = 1/2 \).

The twist number of the vacuum field associated with these corrugations may be approximately calculated by means of (6) (see below).

### 2.2. Twist, Shear, Pressure Profile

Two invariant functions depending on \( s \) are to be prescribed, namely the twist number \( \iota(s) \) and the mass \( m(s) \) within a magnetic surface. The invariant mass function is determined by prescribing an initial pressure distribution \( p(s) \) and assuming throughout \( p = \rho g^2 \). The mass density \( \rho(s) \) and the computed initial volume \( \iota'(s) \) define then \( m(s) \).

The initial profile of the twist number per field period, \( \iota_p \), is represented as a power series in the radius function \( R \):

\[
\iota_p = \mu_0 + \mu_2 R^2 + \mu_4 R^4 + \mu_6 R^6 + \cdots;
\]

\[
i = M \iota_p.
\]

In order to give approximately the twist number \( \iota_p \) of toroidal vacuum fields associated with the given corrugations of the wall, the coefficients of (5) are found either from the corresponding analytical formulas for straight, helically symmetric vacuum fields, (6), or are determined by a fit to the actual \( \iota \)-profile of the toroidal vacuum field whose outermost magnetic surface is approximated (see, for example, Section 3.3). The applied analytical formulas for the twist number \( \iota_p \) for vacuum stellarator fields were derived from [1] and read

\[
\iota_p = \sum_{n=1,2,\ldots} n \left[ \frac{b_n}{B_0} \left( \frac{n}{2} \right) \frac{1}{n!} \right]^2 \left\{ \frac{1 - \frac{1}{n}}{n} E^{n-4} \right\}
\]

\[
+ \frac{n}{2} E^{n-2} + \frac{n^2(2n + 3)}{4^2(n + 1)} E^n + \frac{n^4}{3 \cdot 4^3(n + 1)}
\]

\[
\cdot \left\{ \frac{(n + 3)}{n + 1} \right\} E^{n+1} + \frac{n^6}{4(n + 1)(n + 2)}
\]

\[
\cdot \left\{ \frac{(n + 3)(n + 4)}{4(n + 1)(n + 2)} \right\} E^{n+4} + \cdots
\]
where \( q_n = e \varphi / n, \) \( \varepsilon = 2 \pi b / L_P; \) \( q \) is the minor dimensionless mean radius of a magnetic surface; \( b_n / B_0 \) measures the amplitude of the helical fields related to the main magnetic field \( B_0. \) In the case of a single helical field the twist number per wire period is \( \tau_W = n \tau_P, \) since the wire period is \( n \) times the field period. A few terms are given explicitly in terms of the corrugations \( \Delta_{1ax} \) on axis:

\[
\begin{align}
\tau_1 &= \frac{1}{2} (\varepsilon \Delta_{1ax}^2) \\
\tau_2 &= A_{2ax}^2 \\
\tau_3 &= 2 (\Delta_{3ax} q)^2
\end{align}
\]

(7a) (7b) (7c)

where

\[
\begin{align}
b_1 / B_0 &= 2 \varepsilon \Delta_{1ax}, \\
b_2 / B_0 &= 2 \Delta_{2ax}, \\
b_3 / 2B_0 &= 2 \Delta_{3ax} / \varepsilon, \\
2b_4 / 3B_0 &= \Delta_{4ax} / \varepsilon^2, \quad \varepsilon = R.
\end{align}
\]

In leading orders in \( \varepsilon \) and \( \Delta_1 \) the twist number of helical vacuum fields with the field period \( L_P \) is approximately given by \( \tau_1 = (l-1) A_1^2 \varepsilon^{2(l-2)} \), where for the moment \( \Delta_1 \) is the corrugation of the magnetic surfaces at the radius \( q. \) Since these formulas are applied later to compute nearly current-free equilibria, the agreement (see Fig. 2) between these estimates and the numerically calculated rotational transform was checked for a straight helical \( l = 3 \) field with a rather large corrugation by using a field-line-tracing code; Fig. 3 shows the contours of the magnetic surfaces for \( L_P / 2 \pi b = 1, \) \( \delta_3 = A_3 \varepsilon. \) Analogous studies were done for \( l = 1, 2, \ldots \) fields.

Analytical estimates for the relative depth of the magnetic well depending on these parameters are given in, for example, [1, 15] for systems with a circular magnetic axis (according to [1], p. 183, the depth of the magnetic well in a slender \( l = 2 \) configuration is given by

\[
\frac{11/2 - A_2^2}{} \left( 1 + A_2^2 \right) \left( 1 + A_2^2 \right) > 0
\]

and, for example, [1, 16] for systems with a spatial magnetic axis.

The initial distribution of the pressure \( p \) is a modified Gauss function

\[
p = p_0 \left[ \exp (- \sigma R^2) - \exp (- \sigma) \right] / \left[ 1 - \exp (- \sigma) \right]
\]

(see Fig. 4), where \( \sigma \) defines the compression ratio \( \kappa = b/a; \) the mean plasma radius \( a \) is measured.
Fig. 4. Pressure profile:

\[
p = p_0 \frac{\exp(-\sigma R^2) - \exp(-\sigma)}{[1 - \exp(-\sigma)]}, \quad \sigma = 4.
\]

where the pressure has the \( \exp(-1) \)-fold value of the maximum pressure \( p_0 \). Typical values are \( \kappa = 2 \) \((\sigma = 4)\), \( \kappa = 1.7 \) \((\sigma = 2)\), \( \kappa = 1.3 \) \((\sigma = 0.5)\). Throughout this paper \( \kappa = 2 \) is used if not stated otherwise. In case of \( \sigma = 4 \) and low \( \beta \) the mean \( \langle \beta \rangle \)-value averaged over the volume is approximately \( \langle \beta \rangle = \beta/4 \) of the maximum \( \beta \)-value at the magnetic axis.

3. Numerical Convergence Studies, Results on \( l = 2 \) Stellarators

Convergence studies for the potential energy \( W = \iiint (p + B^2/2) \, dV \) \((V\) is the plasma volume), the axis shift \( \xi \), and the helical \( l = 1 \) deformation of the magnetic axis with mesh size will be reported.

In the following the distance between the mean position of the magnetic axis and the mean geometric centre of the outer wall is called axis shift \( \xi \).

The distance between the mean position of the magnetic axis in finite-\( \beta \) configurations and the mean position of the magnetic axis of the corresponding \( \beta = 0 \) configuration (vacuum field) is called axis displacement \( \delta \) and is given by \( \delta = \xi(\beta) - \xi(0) \), where \( \xi(0) \) is the axis shift for \( \beta = 0 \).

The analytically calculated axis displacement of \[1, 2, 17\] will be compared with the numerical results.

3.1. Convergence of Energy with Mesh Size

Because the physical quantities strongly depend on the discretization errors in three-dimensional computations and because the computer time and storage are limited, extensive numerical convergence studies are required. Relevant results can only be obtained by a convergence study in the grid size. The discretization of the problem was performed on an equally spaced grid of mesh sizes \( \Delta s, \Delta u, \Delta v \). The convergence studies were carried out by varying the mesh sizes simultaneously at constant ratios \( \Delta s : \Delta u : \Delta v \) which define a certain straight line through the origin in the \( (\Delta s, \Delta u, \Delta v) \) parameter space, and, in some cases, by varying each mesh size separately. The dependence of the convergence rate on the mesh size is different in the various directions of the parameter space \( (\Delta s, \Delta u, \Delta v) \) and also depends on the physical parameters.

On a given straight line of convergence through the origin in the \( (\Delta s, \Delta u, \Delta v) \)-space the potential energy \( W \) is represented according to Ref. \[18\] as a power series in \( \Delta s \):

\[
W(\Delta s) = W_0 + W_2(\Delta s)^2 + W_4(\Delta s)^4 + \cdots.
\]

In the particular case of \( W \) \( \text{VII-A} \) parameters the extrapolated value \( W_0 \) and the coefficients \( W_2 \) of Eq. \( (8) \) are determined by solving a weighted least squares approximation problem and read as follows \((g\) is the weight function; the numbers of intervals in \( s, u, v \) directions, \( NS/NU/NV \), are varied from \( 10/10/20 \) to \( 30/30/60 \), \( c = 4, \beta = 2 \cdot 10^{-4} \)): \[
\begin{array}{cccc}
W_0 & W_2 & W_4 & g \\
45.11252 & 38.82 & & \frac{1}{(\Delta s)^2} \\
45.1124 & 41.12 & 6.72 & - 53.97 \\
45.1128 & 40.97 & 6.07 & - 61.10 \\
45.1131 & 40.37 & 17.58 & - 16.31 \\
45.1134 & 40.32 & - 16.27 & - 24.61 \\
\end{array}
\]

Instead of calculating the extrapolated value \( W_0 \) by means of least-squares approximations \((8)\), a local quadratic polynomial \( W_0 + W_2x^2 \) \((x = \Delta s \) for the moment\) was fitted through two neighbouring points \( x_i \), \( x_{i+1} \). The value of \( W_0 \) then depends on the location of the interval \( (x_i, x_{i+1}) \). This series of numbers \( W_0(x_i) \) was extrapolated for \( x \to 0 \) and gives \( W_0 = 45.1113 \). All the extrapolated values for \( W_0 \), which have been determined by means of a polynomial of degree higher than two lie within a relative deviation of the order \( 1 \cdot 10^{-6} \) which is in the order of the accepted accuracy for the energy as obtained by the iteration procedure of the BBG code.
The same procedures have been applied to determine the extrapolated energy \( W_0 \) in a low-\( \beta \), straight \( l=2 \) stellarator equilibrium \( (\beta = 0.004, \Delta_2 = 0.333, 1/\varepsilon = 2.83, \iota_\rho = 0.09395 + 0.00587 R^2) \) with a bell-shaped pressure profile \( p = p_0 (1 - R^2)^2 \).

The iteration procedure to minimize the potential energy is continued until about ten decimal figures remain constant. The effect of the discretization error in simple, straight, helical configurations is substantially smaller than in toroidal configurations. As a result the coefficients \( W_4 \) \((i=2,\ldots)\) of the polynomial (8) are typically three orders of magnitude smaller than in the toroidal \( l=2 \) configuration and read as follows \((c=0, \iota_\rho \text{ to } 45/36/36 \text{ grid})\):

<table>
<thead>
<tr>
<th></th>
<th>( W_0 )</th>
<th>( W_2 )</th>
<th>( W_4 )</th>
<th>( r )</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>0.03318</td>
<td>-0.00044</td>
<td>-0.6609</td>
<td>1 5/2</td>
</tr>
<tr>
<td>2</td>
<td>0.03314</td>
<td>-0.00150</td>
<td>-0.6539</td>
<td>1 3</td>
</tr>
</tbody>
</table>

The extrapolated value of the energy is found to be \( W_0 = 31.6075311 \) with all three above mentioned extrapolation methods and lies within a relative deviation of about \( 3 \cdot 10^{-8} \). This result was compared [19] with \( W_0 \) (TUBE) = 31.6075328, which is computed with the TUBE code; they agree within a relative deviation of about \( 1 \cdot 10^{-7} \). In this case, comparison of convergence studies for the potential energy with different values of the parameter \( c \) \((c=4, 2, 0)\) showed that for \( c=0 \) the convergence rate is more than two orders of magnitude faster than in the case \( c=4 \), i.e. in leading order \( W_2(c=0)/W_2(c=4) \approx 2 \cdot 10^{-3} \).

### 3.2. Toroidal \( l=2 \) Configurations with Small Shear

In the following the \( W \) VII-A stellarator configuration \((M=5)\) is simulated. This is an \( l=2 \) configuration of large toroidal and fairly large helical aspect ratio: \( A = 20, A/M = 4 \). Because of the large helical aspect ratio the shear is small. The single wall parameter used is the elliptical deformation \( \Delta_2 = 0.35 \) \((\Delta_{2ax} = 0.34)\). The spatial dependence of \( i \) is approximately given according to (7b),

\[
[5\Delta_{2ax}^2/(1 + \Delta_{2ax}^2)] \approx 0.53, 0.53 \varepsilon^2/2 \approx 0.02],
\]

by \( i = 0.53 + 0.02 R^2 \), which is in close agreement with the \( \iota \)-profile obtained from vacuum fields by a field-line-tracing code; \( \beta_c = 2 \iota_0^2/A = 0.028; \iota_0 \) and \( \iota_0 \) are the values of \( \iota \) around the torus at the magnetic axis and at the boundary, respectively; the magnetic well is about \( 1\% \) according to Shafranov’s relation (see Section 2.2).

Figure 5 shows the flux surface contours of this stellarator configuration for \( \beta = 0.022 \) \((\iota_\rho = 0.53, \Delta_2 = 0.35, \Delta_{2ax} = 0.53, \iota_0 = 0.55, \beta = 0.022) \). In Fig. 6 the axis shift \( \xi \) is plotted as a function of the mesh size \( \Delta s \) and for various values of \( \beta \).

Because of the torus effect the axis shift \( \xi \) is not zero at \( \beta = 0 \) \((\text{vacuum case})\). The approximate analytical value of \( \xi = \xi = \xi = \xi = 0.04 \) \((\text{marked by an arrow in Fig. 6})\) for \( l = 2, A = 20, M = 5 \). The extrapolated value determined numerically corresponds well to the analytical value.
In finite-β configurations a helical $l = 1$ corrugation of the magnetic surfaces appears [15] besides the axis shift $ξ$ although the boundary has no $l = 1$ corrugation. The helical $l = 1$ deformation of the axis is denoted by $Δ_{1ax}$ and is plotted in Fig. 7 as a function of $Δs$ at various values of $β$. In the case $β = 0$ ($l = 2$ vacuum field) the extrapolated value of $Δ_{1ax}$ vanishes in accordance with asymptotic analytical results [15] and numerical vacuum field computations by a field-line-tracing code. As the plasma pressure increases, the helical $l = 1$ deformation of magnetic surfaces increases with the maximum value at the axis. The period length of this deformation coincides with $L_P$.

Relative to the $l = 2$ corrugation ($Δ_2 > 0$) the phase of the $l = 1$ corrugation is such that at $v = 0$ the magnetic axis is moved closer to the inner edge of the torus (radially inward shift in the plane of standing ellipse), and that at $v = 1/2$ the magnetic axis is moved closer to the outer edge of the torus (i.e. radially outward shift in the plane of lying ellipse; see Fig. 5); this phase corresponds to $Δ_{1ax} < 0$. Consequently, the helical $l = 1$ corrugation of period $L_P$ with $Δ_1 > 0$ can be used to diminish the helical deformation of the plasma column. Results are shown in Fig. 8 for $W$ VII-A parameters. Here the axis displacement $δ$ (solid curve) and the helical $l = 1$ deformation $Δ_{1ax}$ at the magnetic axis (dashed curves) are plotted versus $β/β_c$ for two different helical corrugations of the wall. The case with $Δ_1 = 0$ represents the extrapolated values of the axis displacement in the $W$ VII-A stellarator as discussed above. In the other case the wall has a small helical corrugation $Δ_1 = 0.1$, which shifts the plasma by a small amount to the
main torus axis (i.e. the axis shift of the vacuum field is reduced).

At $\beta = 0$ the $n=1$ deformations of the wall and of the magnetic axis are about the same. Accordingly the rotational transform due to the $l=1$ field is almost uniform in accordance with (7a). The twist number around the torus associated with the $l=1$ deformation $A_1 = 0.1$ is approximately $(A_1 M / A^2 M / 2) \approx 1.6 \cdot 10^{-3}$ and consequently very small compared to $\omega_0 = 0.53$. Therefore the effect of the $l=1$ field on the shift will be small. Increasing $\beta$, the $l=1$ deformation of the axis becomes smaller than that of the wall, so that the effect of the $l=1$ corrugation on the axis shift diminishes. Increasing the $\beta$-value further the phase of the $l=1$ deformation of the axis becomes opposite to that of the boundary. Thus a small helical $l=1$ corrugation of the boundary with appropriate phase and $\beta$-dependent amplitude eliminates the helical deformation $A_{1ax}$ of the axis. This is shown in Fig. 8 (in this case $A_{1ax}$ vanishes at $\beta / \beta_c \approx 0.6$). This result is different from that reported in [20] where it is conjectured that the spiralling can only be removed by a vertical field. Finite-\(\beta\), $l=2$ stellarators with a circular magnetic axis are discussed in, for example, [21] by asymptotic analytical methods.

### 3.3. Short Pitch $l=2, 4$ Configurations of Large Shear

The vacuum twist number of a large shear $l=2, 4$ configuration with $M = 10$ periods and $A = 7.7$ ($M / A = 1.3$, $A_2 = 0.25$, $A_{2ax} = 0.17$,

$$10 A_{2ax} / (1 + A_{2ax}^2) \approx \omega_0 = 0.28, \omega_b = 0.79$$

is approximated according to (6) by

$$\omega = 0.28 + 0.16 R^2 + 0.35 R^4.$$
in \( l = 2 \) configurations the shear reduces the axis displacement not by one order of magnitude but by a small amount. In contrast to results from asymptotic theories [3] an estimate of the maximum equilibrium \( \beta_e \)-value is of the order \( \beta_e = 2 \omega_b^2/A = 0.02 \) and not of the order \( 2 \omega_b^2/A = 0.16 \). In fact, from Fig. 11 it is found that the maximum equilibrium \( \beta_e \)-value is \( \beta_e \approx 0.03 \) (if the admissible shift is bounded to \( \xi \approx 0.5 \)); this value is somewhat higher than the estimate \( \beta_e = 0.02 \).

In Fig. 12 these numerical results on the axis displacement are compared with analytical results derived by Shafranov ([1], p. 130) for \( l = 2 \) stellarators under the conditions \( 1/A \ll 1, M/A \ll 1 \) (hence \( \beta_e \approx \beta_c [1 + (M/A)^2/2 + \cdots]^2 \approx \beta_c \) for \( l = 2 \) configurations):

\[
\delta_{sh} = \frac{\beta_c}{3 \beta} \left\{ -1 + \left[ 1 + 3(\beta/\beta_c)^2 \right]^{1/2} \right\}
\approx \frac{\beta}{2 \beta_b} \left[ 1 - 3 \beta/4 \beta_b + \cdots \right] \tag{10}
\]

(dashed curve in the left graph of Figure 12). In leading order in \( \beta \) the displacement is linearly proportional to \( \beta/\beta_b \) (solid curve in the left graph of Fig. 12) and agrees well with the numerical result for the slender \( l = 2 \) configuration.

Recently, Strauss et al. [17] calculated the displacement including leading order terms due to the finite helical aspect ratio:

\[
\delta_{SM} = (\beta/2 \beta_b) \left[ 1 + (M/A)^2/2 \right]/\left[ 1 + (M/A)^2/4 \right] \tag{11}
\]

which corresponds in leading order to (10) by neglecting terms of order \( (M/A)^2 \). (The factor \( 1 + (M/A)^2/2 \) also appears in Shafranov's formula including next order terms.) A graph of (11) is shown in Fig. 12 for \( M/A = 1.3 \) (solid line in the right picture; the dashed straight is Shafranov's result \( (\beta/2 \beta_b) [1 + (M/A)^2/2] \) including next order terms in \( (M/A) \)). It is found that the axis displacement obtained numerically (a) depends distinctly on the helical aspect ratio \( A/M \), (b) agrees with results of [1, 17] in leading order in \( \beta \) for small values of \( \beta \) and \( M/A \); hence the numerically determined axis displacement scales in the helical aspect ratio similarly to that of [1, 17] for small values of \( M/A \) and similarly to that of [1] for large values of \( M/A \). The deviations in Fig. 12 between the numerical and the analytical results may be due to the fact that the asymptotic theories are beyond their applicability for these parameters.

### 3.4. Helical \( l = 1 \) Deformation of Magnetic Axis for Various \( l = 2 \) Equilibria

Simultaneously with the displacement due to the finite beta, a helical \( l = 1 \) deformation of the plasma appears (see Figure 5). The extrapolated values of the axis displacement versus \( \beta/\beta_b \) in \( l = 2 \) (left picture) and \( l = 1, 2 \) (right picture) stellarators compared with asymptotic theories.
Fig. 13. Extrapolated values of the helical \( l = 1 \) axis deformation \( \Delta_{1ax} \) in small and high shear \( l = 2 \) stellarators.

\((\Delta \delta \to 0)\) of the \( l = 1 \) deformation \( \Delta_{1ax} \) of the magnetic axis are plotted in Fig. 13 as a function of \( \beta/\beta_b \) for (a) the \( W \) VII-A stellarator with small shear (circle), (b) the short pitch \( l = 2, 4 \) stellarator discussed above (squares), and (c) are compared with asymptotic theories. From the analytical theory by Shafranov et al. [15], taking into account leading order terms in \( \beta \) and \( \Delta_{2ax} \), it follows for \( \Delta_{1ax} \) that

\[
\Delta_{1ax}(\beta) - \Delta_{1ax}(\beta = 0) \\
\approx (\beta/2\beta_c) \Delta_{2ax}(1 - \Delta_{2ax}^2)/(1 + \Delta_{2ax}^2) \\
\approx (\beta/2\beta_c) \Delta_{2ax} + O(\Delta_{2ax}^3) \approx \delta \cdot \Delta_{2ax} \\
\approx \frac{\beta}{2\beta_b} [1 + (M/A)\beta^2/2]^2 \Delta_{2ax}.
\]

In this ordering the helical deformation \( \Delta_{1ax} \) is a linear function of \( \beta \) and of \( \Delta_{2ax} \), and is proportional to the displacement \( \delta \). Figure 13 shows that for \( l = 2 \) stellarators all values of the helical deformation \( \Delta_{1ax} \) are close to the Shafranov curve, (12), (solid straight in the figure). Consequently, for \( W \) VII-A parameters \((M/A = 1/4, \Delta_{2ax} = 0.34)\) the helical \( l = 1 \) deformation of the magnetic axis is about 1/3 of the axis displacement.

The approximately linear dependence of both the \( l = 1 \) deformation \( \Delta_{1ax} \) and the displacement \( \delta \) on \( \beta \) has also been found numerically in \( l = 1, 2 \) and \( l = 0, 1, 2, 3 \) configurations, as can be seen from Figs. 8 and 16. At sizable \( \beta \)-values the phase of the \( l = 1 \) deformation of the magnetic axis may be opposite to that of the boundary.

4. Configurations with \( l = 0, 1, 2, 3 \) Fields

Results on a particular \( l = 0, 1, 2, 3 \) configuration are reported here which qualitatively simulates Advanced Stellarator equilibria as reviewed in [14]. The study was also partly motivated by the result reported in Refs. [22—26] on the relation between the negative \( \gamma'' \) property and the reduced axis shift.

The magnetic surface contours of an equilibrium with sizable shear are shown in Fig. 14 \((\beta = 0.15\) corresponds to \( \langle \beta \rangle = 0.04 \) because of the rather peaked pressure profile). The parameters of the configuration and the dependence of the axis shift on the mesh size \( \Delta s \) is shown in Figure 15. The corrugations corresponding to \( l = 0, 1, 2, 3 \) fields all have sizable amplitudes. The twist number is approximated according to (6) by \( \iota = M_{lp} = 0.4 + 0.5 \beta^2 \); the value for \( \beta_c = 2\beta_c^2/A = 0.032 \) is rather small. Nevertheless the permissible \( \beta \) value for an equilibrium with almost no axis shift is about
β = 0.15 (corresponding to $\langle \beta \rangle = 0.04$) and consequently a factor of about five higher than the above estimate for the critical $\beta_c$-value. The displacement of the plasma column is almost independent of beta in the range of $\beta \lesssim 0.12$, as can be seen in Figure 16. The helical excursions of the wall and the magnetic axis are almost the same. Hence this high shear $l = 0, 1, 2, 3$ configuration is characterized by a higher magnetic stiffness as compared to the small shear and the high shear $l = 2$ stellarators.

If the $\beta$-value is increased to $\beta = 0.19$, the shift $\xi$ increases nonlinearly with decreasing mesh size such that no extrapolation to zero mesh size could be performed. This singular behaviour could be an indication of instability (e.g. unstable rigid motion of the plasma in the radial outward direction) and of non-existence of an acceptable equilibrium. When the $\beta$-value is increased further, the extrapolation of the axis shift to zero mesh size can be performed; but the magnetic axis is very close to the wall and hence these configurations are of no interest. It has not yet been investigated whether the singular behaviour of the axis shift in this intermediate $\beta$-range is a hint for bifurcation or is akin to that found by Greene et al. [2], who showed that at a certain $\beta$-value an equilibrium does not exist in their asymptotic theory.

The results of the present study are that (a) in the case $\beta = 0$ the magnetic axis is shifted in radial inward direction with respect to the geometric centre of the wall, (b) an almost centred equilibrium is obtained at $\beta = 0.13$ ($\langle \beta \rangle = 0.033$) (in contrast to pure $l = 2$ stellarator equilibria where centred equilibria could not be found) (c) the axis displacement is rather small for $\beta \lesssim 0.12$, and (d) a positive value $\Delta > 0$ shifts the plasma in radial inward direction. The extrapolated values of the axis displacement $\delta$ and the helical axis deformation $\Delta_{\text{ax}}$ are shown in Fig. 16 as functions of $\beta$.

A similar mechanism of producing toroidal stellarator equilibria was used in high-beta stellarator experiments [10, 11] which are characterized by a large aspect ratio.

5. Simulation of the W VII-AS Configuration

The contours of the magnetic surfaces of the W-VII-AS configuration as taken as initial and boundary values for the BBG code are shown in Fig. 17 (left column); the vacuum field configuration computed by a field-line-tracing code as given in [27] is practically identical. The right column of Fig. 17 shows the magnetic surface contours as obtained by the BBG code for $\beta = 0.045$ ($p = p_0 [1 - R^2]^2, \gamma = 0.008 R^2, A = 9, M = 5, \langle \beta \rangle = 0.03$). As can be seen from this figure, the axis shift at $v = 0$ (standing ellipse) is about half the minor half-axis. A larger shift than this value does not seem acceptable in this particular configuration. Hence the critical equilibrium beta value at the magnetic axis is about $\beta_c = 0.045$. The extrapolated values of the axis shift are shown in Fig. 18 as a function of $\beta$. 
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\[
p = p_0 \left(1 - R^2\right)^2, \ i = 0.38 \left(1 + 0.008 R^2\right), \ \beta_c = 0.032, \ A = 9, \ M = 5, \ A_0 = -0.09, \ A_1 = 0.3, \ A_2 = -0.06, \ A_3 = 0.03, \ A_{22} = -0.02, \ A_{32} = 0.04, \ A_{33} = -0.06, \ A_{41} = 0.01, \ A_{42} = -0.01, \ A_0 = 0, \text{ where the boundary is given by}
\]

\[
r_b(u, v)/b = (1 - A_0 \cos V) \cos U + A_1 \cos V - \sum A_{lm} \cos \left[(l - 1)U - mV\right],
\]

\[
z_b(u, v)/b = (1 - A_0 \cos V) \sin U + A_1 \sin V + \sum A_{lm} \sin \left[(l - 1)U - mV\right].
\]

In order to display by a finite mesh size the shift of the plasma column according to the extrapolated curve of Fig. 18, a somewhat higher \(\beta\)-value has been used such that the axis shift has got the extrapolated value \(\xi = 0.35\).

Although the optimization of the \(W\ VII\)-AS stellarator was strongly limited by accounting for the hardware restrictions imposed by the existing \(W\ VII\) machine, the improvement in the accessible equilibrium beta is a factor of about two.

6. Conclusions

Toroidal non-symmetric solutions of the magnetohydrostatic boundary-value problem were computed with the BBG code. Since expansion in a small parameter is not applied in the code, the results are valid for arbitrary \(\beta\) and arbitrary finite deformations of the magnetic surfaces and of the magnetic axis. Detailed numerical convergence studies with respect to the mesh size turned out to be indispensable.

It has been shown that not only in the \(l = 2\) stellarator with almost no shear but also in the short pitch \(l = 2\) stellarator with large shear, acceptable equilibrium beta values are not accessible for moderate values of \(i\). Consequently advanced stellarator configurations of moderate aspect ratio have to be discussed to increase the equilibrium beta to reasonable values.

In advanced \(l = 0, 1, 2, 3\) stellarator configurations with reduced Pfirsch-Schlüter currents the radially outward displacement of the magnetic axis is a factor of at least three smaller than in pure \(l = 2\) stellarators. The finite-\(\beta\) calculations showed that these configurations have a magnetic stiffness much higher than in pure \(l = 2\) stellarators. Hence the critical equilibrium-\(\beta_e\) value can be increased by at least a factor of three compared with the pure \(l = 2\) stellarator. The \(W\ VII\)-AS stellarator is a step in this direction.

Fig. 18. Extrapolated values of the axis shift versus \(\beta\) for the \(W\ VII\)-AS configuration (the parameters are given in Figure 17).
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