Stellarator Expansion at Finite Aspect Ratio

D. Lortz and J. Nührenberg
Max-Planck-Institut für Plasmaphysik, EURATOM-Association, Garching

Z. Naturforsch. 37a, 876—878 (1982); received April 6, 1982

To Professor Arnulf Schlüter on his 60th Birthday

A previous stellarator expansion valid for finite aspect ratio and small rotational transform is based on the axisymmetric toroidal vacuum field as zeroth order field. The equilibrium beta value behaves as $\beta \sim \varepsilon^2$. The condition for the magnetic surfaces to be unaffected by $\beta$ within this ordering is formulated.

I. Introduction

Various stellarator expansions have hitherto been obtained for large aspect ratio by many authors (see, e.g. [1—3]). Recently, a low-$\beta$ stellarator expansion at finite aspect ratio was devised [4], which was based on the axisymmetric toroidal vacuum field as zeroth order field. Here, we generalize this type of expansion to arbitrary closed line vacuum fields as zeroth order field. A prerequisite for the equilibrium expansion is the asymptotic expansion of magnetic surfaces at small values of the rotational transform [5, 6] which has been obtained in an explicit form [7] lending itself to application. Stellarator expansions at finite aspect ratio should be particularly suitable for the investigation of separatrix formation and its relation to the $\beta$-value [8, 9]. Arbitrary closed line vacuum fields as zeroth order are necessary for an adequate description of toroidal equilibria with significantly reduced parallel current density [10, 11].

II. Equilibrium Expansion for Small Rotational Transform

The MHD equilibrium equations are written in the form

\[ \nabla \cdot B = 0 , \]  
\[ B \cdot \nabla F = 0 , \]  
\[ B \cdot \nabla j = (dp/dF) \nabla \cdot (\nabla F \times B/B^2) , \]  
\[ j = \nabla \times B \]  

where $F$ describes the magnetic surfaces, $j = j \cdot B/B^2$ is related to the parallel current density; (3) guarantees that $j$ is divergencefree, so that (4) is integrable.

The following ordering is employed

\[ B = B_0 + \varepsilon B_1 + \varepsilon^2 B_2 + \cdots , \]  
\[ F = F_0 + \varepsilon F_1 + \varepsilon^2 F_2 + \cdots , \]  
\[ \iota \sim 0 (\varepsilon^2) , \]  
\[ \beta \sim \varepsilon^2 \]  

where $\iota$ is the rotational transform and $\beta$ characterizes the ratio of thermal to magnetic energy.

$B_0$ is taken as a vacuum field with toroidally closed lines, so that it may be represented as

\[ B_0 = \nabla \psi = \nabla \psi \times \nabla \chi \]  

with single-valued functions $\psi$ and $\chi$. In the following, $\psi$, $\chi$, and $\varphi$ are used as independent variables (coordinates). The following relations hold

\[ D = (\nabla \psi \times \nabla \chi) \cdot \nabla \psi = B_0^2 = 1/|\psi|, \]  
\[ B_0 = D r, \quad B_0 \cdot \nabla = D \partial_\varphi, \]  
\[ g_0^{\psi \varphi} = g_0^{\psi \varphi} = g_0^{\varphi \varphi} = 0 . \]  

The first and higher order fields are represented as in [7] by

\[ B_r = \nabla u_r \times \nabla \psi - \nabla v_r \times \nabla \chi , \quad r \geq 1 . \]  

Since we are interested in a stellarator expansion $B_1$ is the curlfree leading order stellarator field with single-valued $u_1$ and $v_1$ in accordance with (5). With (6) and (8), (1) is automatically satisfied, i.e. $B$ divergencefree. In $\psi$, $\chi$, $\varphi$ coordinates

\[ B_r \cdot \nabla = D[v_r, \varphi] \partial_\psi + u_r, \varphi \partial_\chi \]  
\[ - (u_r, \chi + v_r, \psi) \partial_\varphi \]  

Reprint requests to Dr. D. Lortz, Max-Planck-Institut für Plasmaphysik, D-8046 Garching.

0340-4811 / 82 / 0800-0876 $ 01.30/0. — Please order a reprint rather than making your own copy.
is obtained, so that (2) reads
\[ \partial_{\varphi} F_0 = 0, \]
\[ \partial_{\varphi} F_\mu + \mu \sum_{r=1} v_r, \varphi \partial_{\varphi} + u_r, \varphi \partial_{x} \]
\[ - (u_{r,x} + v_r, \varphi) \partial_{\varphi} | F_{\mu - r} = 0. \] (10)

The results
\[ F_0 = F_0(\varphi, \chi), \]
\[ F_1 = j_1 + g_1, \]
\[ f_1 = - v_1 F_{0, \varphi} - u_1 F_0, \chi, \]
\[ g_1 = g_1(\varphi, \chi) \] (11)
are easily obtained [7].

Since \( B_0 \) and \( B_1 \) are curlfree, the Eqs. (5) are
supplemented by
\[ j = e^2 j_2 + e^3 j_3 + \cdots, \]
\[ j = e^2 j_2 + e^3 j_3 + \cdots, \] (12)
so that the leading order of (3) reads
\[ \partial_{\varphi} j_2 = 0 \]
with the result
\[ j_2 = j_2(\varphi, \chi). \] (13)

The third order of (3) yields
\[ \partial_{\varphi} j_3 + u_1, \varphi j_2, x + v_1, \varphi j_2, \varphi = 0, \]
so that
\[ j_3 = - u_1 j_2, x + v_1 j_2, \varphi + j_3(\varphi, \chi). \] (14)

Because of the ordering of \( \beta \), the fourth order of (3) is obtained as \( (\partial \beta = \partial_{\varphi} / \partial F_0) \)
\[ D \partial_{\varphi} j_4 + B_1 \cdot \nabla j_3 + B_2 \cdot \nabla j_2 = D p' D^{-1}, \]
where \( D = F_{0, \varphi} \partial_{\varphi} - F_0, \varphi \partial_{x} \) differentiates parallel
to \( F_0 = \text{const} \), i.e. poloidally.

Its solubility condition \([\cdots] = \int d\varphi / D(\cdots)\]
\[ \langle B_1 \cdot \nabla j_3 + B_2 \cdot \nabla j_2 \rangle = p' \langle D D^{-1} \rangle \]
\[ = p' D Q = D p' Q, \] (15)
where \( Q = \int d\varphi / D = \int d\varphi / B_0 \) and the integral is performed
along the zeroth order field lines, is the leading order equilibrium equation.

The evaluation of (15) is performed in two steps. First,
\[ \langle B_1 \cdot \nabla j_3 \rangle = \langle D (j_2, \varphi \partial_{x} - j_2, x \partial_{\varphi} (u_1 v_1, \varphi) \rangle \]
is obtained with the help of (14). Second, the flux of the second order field \( B_2 \) through two lines of the
zeroth order field is introduced
\[ U = \int B_2 \cdot d\varphi / \partial \]
\[ = \int D^{-1} B_2^\varphi d\varphi d\chi - \int D^{-1} B_2 \partial_{\varphi} d\varphi d\chi \] (16)
to compute
\[ - j_2, x U, \varphi + j_2, \varphi U, x = \langle B_2 \cdot \nabla j_2 \rangle. \]
Now the result [7]
\[ - U - \int u_1, \varphi d\varphi = H(\varphi, \chi) \] (17)
is employed, i.e. that this combination only depends on \( F_0 \) (for the interpretation of \( H \) as second order
poloidal flux, see Appendix). Because of
\[ (j_2, \varphi \partial_{x} - j_2, x \partial_{\varphi}) H = H' D j_2, \]
(15) may be integrated to give
\[ j_3 = \langle H(\varphi, \chi) - Q p' d\varphi / dH \rangle. \]

The structure of this equilibrium problem may be elucidated by the following iteration scheme
\[ F_0(\varphi, \chi) \Rightarrow \hat{j}_3(\varphi, \chi) \Rightarrow B_2 \Rightarrow U \Rightarrow F_0, \] (19)
where the first step is accomplished by (18) [for given \( p' d\varphi / dh \) and \( j_2(\varphi, \chi) \)]; the second step involves solving
(4), e.g. by obtaining the vector potential of \( B_2 \) via
Poisson’s integral with \( j_2 \) as kernel; the third and
the fourth step are explicitly given by (16) and (17). While the second step in general is a three-dimen-
sional problem, it becomes two-dimensional in special cases, e.g. if the zeroth order field is the axi-
symmetric toroidal vacuum field, see [4].

With regard to stellarators, the case of vanishing
net toroidal current \( J \) through each magnetic surface
is of special interest. In leading order
\[ J = e^2 J_2 = e \int j_2 d\varphi d\chi, \]
so that this current can be made zero by appropriate
choice of \( J_2(H) \) in (18). In an iterative procedure
following (19) this would be part of the first step.

**III. Discussion**

In the expansion set forth above the equilibrium
beta value behaves as
\[ \beta \sim 0(e^4) \sim \epsilon^2. \] (19)

An interesting special case occurs if
\[ Q = Q(F_0, \text{vac}), \] (20)
where \( F_0, \text{vac} \) describes the zeroth order magnetic
surfaces as obtained from the vacuum stellarator fields $B_1$ and $B_2$. Equation (18) shows that $j_2$, i.e. the leading order parallel current density, then vanishes identically if $j_2$ does. Thus, the zeroth order magnetic surfaces are unaffected by $\beta$ [within the ordering (19)] if (20) holds. In particular, there then is no Shafranov shift in accordance with the results obtained with 3D codes [10, 11] for vacuum fields obeying a relation similar to (20).

Another interesting consequence arises with respect to MHD stability if (20) is satisfied. Normally, the stability behaviour of a stellarator is intricate even with the orderings used, because the magnetic well and other terms in the stability criteria (see, e.g. [12]) depend on $\beta$, since $F_0$ depends on $\beta$. If (20) holds, the stability behaviour, for $\beta$ given by the ordering (19), is completely determined by the vacuum magnetic field. In particular, stability holds [12] if there exists a vacuum magnetic well ($\Phi, \Phi > 0$, $\Phi$ longitudinal flux, $\ldots = d/dV$, $V$ volume of zeroth order surfaces $F_0$).

Finally, one may ask whether or not (20) allows to order $\beta$ larger than has been done here. An obvious conjecture is $\beta \sim 0(1)$, which, however, would not allow to start the expansion from a vacuum field, see (6), but only from finite-$\beta$ equilibria with zero transform. In such equilibria the surfaces $Q = \text{const}$ are determined and no freedom would be left to make them coincide with the zeroth order magnetic surfaces, see (17). So, the possibility of an expansion with $\beta \sim 0(1)$ seems unlikely. Future work investigating this problem more closely will use an ordering in which $\beta$ and $\varepsilon$ are small but independent [in contrast to (5)] if (20) holds.

Appendix

Here, we prove that (17), viz.

$$U + \int v_1 u_1, \varphi, d\varphi$$

(21)
is the leading order poloidal flux. First, the equations for the perturbed field lines

$$\psi = \psi + \varepsilon \psi_1 + \varepsilon^2 \psi_2 + \cdots = \text{const},$$

$$\chi = \chi + \varepsilon \chi_1 + \varepsilon^2 \chi_2 + \cdots = \text{const}$$

are obtained from

$$B \cdot \nabla \psi = 0,$$

$$B \cdot \nabla \chi = 0,$$

via the representation of $B_\varphi, \varphi \geq 1$, see (8). We obtain

$$\psi_1 = -v_1,$$

$$\chi_1 = -u_1,$$

$$\psi_2, \varphi = -v_2, \varphi + u_1, \varphi v_1, \chi - u_1, \chi v_1, \varphi,$$

$$= -D^{-1} B_2 \varphi + u_1, \varphi v_1, \chi - u_1, \chi v_1, \varphi,$$

$$\chi_2, \varphi = -u_2, \varphi + v_1, \varphi u_1, \chi - v_1, \varphi u_1, \varphi,$$

$$= -D^{-1} B_2 \chi + v_1, \varphi u_1, \psi - v_1, \varphi u_1, \psi,$$

so that the displacement of the field lines after one toroidal turn $\varphi + \Delta \varphi$ is given by [see (16)]

$$\psi_2 (\varphi + \Delta \varphi) = -\partial_\chi (U + \int v_1 u_1, \varphi d\varphi),$$

$$\chi_2 (\varphi + \Delta \varphi) = \partial_\psi (U + \int u_1 v_1, \varphi d\varphi).$$

Second, we take $\psi$ as label for the zeroth order magnetic surfaces and $\chi$ as poloidal variable. Then $\psi_2 \equiv 0$ and the poloidal flux can be obtained as the flux of the zeroth order field through the second order band (at $\varphi = \text{const}$) which is given by the starting line of field lines at $\chi = \text{const}$ and the image line $\chi + \varepsilon^2 \chi_2$ obtained after one toroidal turn:

$$\int B_0 (r, \varphi, r, \chi) d\varphi d\chi = \int d\chi d\varphi = \varepsilon^2 \int \chi_2 d\varphi$$

$$= \varepsilon^2 (U + \int u_1 v_1, \varphi d\varphi).$$