Individual Peeling of Multiple Singular Points

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It is shown that certain systems may exhibit multiple generating singular points which individually peel to result exploded points or limit cycles.

It was recently shown that the concept of exploded points, i.e. the objects of zero f-dimension, leads to detecting possible objects resulting from bifurcation of a generating singular point [1]. An illustrative example given was the Lorenz equation. The characteristics of an exploded point are that the observed object can not be a periodic orbit such as a limit cycle which is of f-dimension one.

An exploded point may not, however be the result of a bifurcation. Its existence may be possible in nonbifurcating systems. An exploded point as any other singular point may be either stable or unstable, or intuitively, attracting or repelling the trajectories in the surrounding neighborhood. If $X$ (dim $X = n$) is the space of interest, an exploded point $x_e \in X$ and taking $x_e$ as the zero f-dimensional subspace of $X$, $x_e \subset X$, dim $x_e = 0$, dim $X = n$.

If a system goes through some bifurcations and results in some "chaotic objects", by the technique of detecting an exploded point, the dimension and the nature of the chaotic object may be determined. The bifurcation analysis of the Lorenz system, (1), given as an example for exploded points in [1] is discussed in detail in [2]. It is also noted in [1] that the Lorenz system cooperatively peels [4] which is an important concept that will be used in analyzing the objects appearing as a result of bifurcations.

In this paper we refer to two more examples (2) and (3) taken from [5].

(1) Lorenz
\[
\begin{align*}
\dot{x} &= -mx + my, \\
\dot{y} &= -xz - r - y, \\
\dot{z} &= xy - bz,
\end{align*}
\]

(2) Screw-Type
\[
\begin{align*}
\dot{x} &= -y - z, \\
\dot{y} &= x + ay, \\
\dot{z} &= b + xz - cz.
\end{align*}
\]

The bifurcation analysis of (2) is given in [6]. Referring to Table 1 in [6] we can see that for $a < \sqrt{2}$, and $c > 2a$, the decomposed peeling of the stable generating singular point yields eigenvalues for the two bifurcated points such that
\[
1 w_{bs}^1 \oplus 1 w_{bs}^1 \oplus 2 w_{bs}^1,
\]
yielding collectively $2w_{bs}^1$. Thus the object for the screw-type chaos has the dimension $2 - 1 = 1$, a stable limit cycle is expected. In fact in [6] it is detected that a limit bundle exists which is basically a limit cycle with a very long period, see Fig. 2 in [6].

We will now discuss the third system, (3) and conclude some interesting results exhibited by this system.

The equations are taken as (3) i, ii, and iii. Here the parameters are $a$, $b$, $c$, and $d$. The singular solutions are found by setting $\dot{x} = \dot{y} = \dot{z} = 0$ to yield
\[
\begin{align*}
y &= x^2/a, \\
z &= (bx + d)/c, \\
x^3 + a \left( \frac{b}{c} - 1 \right) x + \frac{ad}{c} &= 0.
\end{align*}
\]

Denoting
\[
\begin{align*}
A &= a \left( \frac{b}{c} - 1 \right), \\
B &= ad/c,
\end{align*}
\]
and noting that since the coefficient of the $x^2$ term in (6) is zero
\[
x_1 + x_2 + x_3 = 0,
\]
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where x's are either real or imaginary, the following cases for the solutions of Eq. (6) are found, see Table 1, and Figure 1.

The change in the number of singular solutions takes place when B crosses $B_2^*$ and $B_1^*$. To find the values of $B_1^*$ we notice that in Fig. 1 the slope of (6) must vanish at $P_1$ and $P_2$, thus differentiating (6) with respect to x we obtain

$$x = \pm \left[ \frac{1}{3} a \left( 1 - \frac{b}{c} \right) \right]^{1/2}$$  \hspace{1cm} (10)

or in terms of $A$,

$$x = \pm \left[ - \frac{A}{3} \right]^{1/2}.$$  \hspace{1cm} (10')

This shows that this case is possible only in the case of $A < 0$ as it can be seen in Figure 1b. Substituting the x-values from (10) in (6) one obtains,

At $P_1$: $x_{p1} = - \left( \frac{-A}{3} \right)^{1/2}$,

$B_1^* = \frac{2}{3} A \left( - \frac{A}{3} \right)^{1/2}$.

At $P_2$: $x_{p2} = + \left( \frac{-A}{3} \right)^{1/2}$,

$B_2^* = - \frac{2}{3} A \left( - \frac{A}{3} \right)^{1/2}$.

From (11), by substituting (7) for $A$ and (8) for B's

At $P_1$: $d_1^* = \frac{2}{3} (b - c) \left[ - a \left( b/c - 1 \right)/3 \right]^{1/2}$, \hspace{1cm} (13)

Table 1. Real solutions of Equation (6) for various $A$ and $B$ values.

<table>
<thead>
<tr>
<th>$A$</th>
<th>$B_2^*$</th>
<th>$B_1^*$</th>
<th>$B_1^*$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$B_2^* &gt; 0$</td>
<td>$B_2^* = B_2^*$</td>
<td>$B_2^* &gt; 0$</td>
<td>$B_2^* &gt; 0$</td>
</tr>
<tr>
<td>$&lt; 0$</td>
<td>$x_1 &lt; 0$</td>
<td>$x_2 &gt; 0$</td>
<td>$x_3 &gt; 0$</td>
</tr>
<tr>
<td>$= 0$</td>
<td>$x_1 &lt; 0$</td>
<td>$x_2 = x_3 &gt; 0$</td>
<td>$x_3 &gt; 0$</td>
</tr>
<tr>
<td>$&gt; 0$</td>
<td>$x_1 &lt; 0$</td>
<td>$x_2 = x_3 &gt; 0$</td>
<td>$x_3 &gt; 0$</td>
</tr>
</tbody>
</table>

Using Equation (9),

At $P_1$: $x_1 = x_2 < 0$ and $x_3 > 0$, $-2x_1 = x_3$.

At $P_2$: $x_1 < 0$ and $x_2 = x_3 > 0$, $x_1 = -2x_3$.

Dividing (6) by $(x - x_3)$, we obtain

$$x^2 + x_3 x + (A + x_3^2) = 0.$$  \hspace{1cm} (15)

Solving this quadratic equation, it is found that

$x_{1,2} = \left[ -x_3 \mp \left( -4A - 3x_3^2 \right)^{1/2} \right]/2$.

Evaluating this result at $P_1$ and $P_2$ one finds,

At $P_1$: $(x_1 = x_2)$, $x_1, 2 = \left( -A/3 \right)^{1/2}$,

$x_3 = 2 \left( -A/3 \right)^{1/2}$.

Table 1. Real solutions of Equation (6) for various $A$ and $B$ values.
At \( P_2: (x_2 = x_3), \quad x_1 = -2(-A/3)^{1/2}, \)
\[ x_{2,3} = \mp (-A/3)^{1/2}. \]

The Hessian for the system (3) can easily be formed as
\[
\Lambda = \begin{vmatrix}
(1 - y) & -x & -1 \\
2x & -a & 0 \\
b & 0 & -c
\end{vmatrix}
\]
\[ = (1 - y)ac - 2cx^2 - ab. \quad (16) \]

Evaluating (16) at a solution where \( y = x^2/a \)
(Eq. (4)),
\[ \Lambda = (1 - 3y)ac - ab \]
\[ = a[(1 - 3y)c - b]. \quad (16') \]

From Table 1 it can be seen that a change in the number of singular solutions take place for either \( A = 0 \), or \( B = B_1* \) or \( B_2* \). These two cases yield the following results.

**Case I:**

\( A = 0 \) is possible if either \( a = 0 \), or \( b = c \)
(see (7)).

\( a = 0 \) yields \( \Lambda = 0 \).

\( b = c \) yields \( \Lambda = -3ayb \) or \(-3xb^2 \)
where \( b \) can not vanish, see (6),
\( y \) vanishes implies \( x = 0 \), in
\( \text{turn either } a \text{ or } d = 0. \)

Thus \( d = 0 \) yields \( \Lambda = 0. \) (Notice that \( B = 0. \))

**Case II:**

\( B = B_1* \) or \( B_2* \) is possible, see (11) and (12).

\( a = 0 \) is the same result as above,
\( (1 - 3y*)c - b = 0 \) must be evaluated at the singular points.

Referring to (4), and the solutions obtained from
(15).

At \( x_{1,2} \)
\( (1 + A/a)c - b = 0 \) is satisfied for all \( a, b, c. \)

At \( x_3 \) the result is \( b = c. \)

For \( P_2 \) the same values are found for the double point \( x_2, x_3 \) and the single point \( x_1 \) corresponding to the \( x_1, x_2 \) and \( x_3 \) above, respectively.

The stability consideration at singular points is based on the linearized equations about these points where the characteristic equations become,
\[
\begin{vmatrix}
(1 - y*) - s & -x* & -1 \\
2x* & -a - s & 0 \\
b & 0 & -c - s
\end{vmatrix} = 0
\]

which yields
\[ -s^3 + (1 - a - c - y)s^2 + [a(1 - c) + (c - b) - y(c + 3a)]s \]
\[ + a[(c - b) - 3cy] = 0. \quad (17) \]

Based on the analysis of the number of singular solutions, Table 1, and the parameter values where the Hessian vanishes we can arrive at the conclusion that
\[ a = 0, \quad b/c = 1, \quad d = 0 \quad (18) \]
are candidates to be considered in the bifurcation analysis. Studying the stability properties of the generating solutions for various combinations of these parameters we see that the system has one stable and one unstable part as in the case of the Lorenz system, [3].These cases are summarized below:

**Stable system:**
\[ a > 0, \quad b/c < 1, \quad \text{and } d > 0; \]
\[ a < 0, \quad b/c > 1, \quad \text{and } d < 0; \]

**Unstable system:**
\[ a > 0, \quad b/c < 1, \quad \text{and } d < 0; \]
\[ a < 0, \quad b/c > 1, \quad \text{and } d > 0. \]

These regions are shown in Figure 2.

**Discussion and Conclusions**

As it was done for the systems (1) and (2), the third system can also be completely studied via the bifurcation analysis qualitatively. The following conclusions are significant.

<table>
<thead>
<tr>
<th>Change in Number of Sing. Sol.</th>
<th>( A = 0 )</th>
<th>Stability Changes</th>
<th>Bifurcation Parameter</th>
</tr>
</thead>
<tbody>
<tr>
<td>( a = 0 )</td>
<td>Yes</td>
<td>Yes</td>
<td>Yes</td>
</tr>
<tr>
<td>( b/c = 1 )</td>
<td>Yes</td>
<td>Yes</td>
<td>Yes</td>
</tr>
<tr>
<td>( d = 0 )</td>
<td>Yes</td>
<td>Yes</td>
<td>No</td>
</tr>
</tbody>
</table>

Here \( d \) basically determines the global stability of the system, Figure 2.

**Conclusion 1:** If \( a \) and \( b/c \) values are fixed, for \( d > 0 \) (or < 0) the system is globally stable (or unstable), \( d \) is not a bifurcation parameter.
The generating singular point, \( x_1 \), for the stable system has eigenvalues as follows:
\[
\begin{align*}
s_1 < 0, & \quad \text{Re} \ s_2 < 0, \quad \text{Re} \ s_3 < 0
\end{align*}
\]
(Stable focus).

On the other hand, that of the unstable system possesses,
\[
\begin{align*}
s_1 > 0, & \quad \text{Re} \ s_2 < 0, \quad \text{Re} \ s_3 < 0.
\end{align*}
\]
(Saddle-focus).

We are interested in the stable system. There are two successive bifurcation phenomena, see Table 1. The first one is the creation of a second generating point:
\[
x_1 \rightarrow (x_1, x_2, 3)
\]
which takes place at \( B_2^* \) value. The characteristic values are:

Before: \( x_1; \quad 3 w_{m_1}, \)

After: \( x_1; \quad 2 w_{m_1} + 1 w_{m_2}, \)
\( x_2, 3; \quad 1 w_{m_1} + 2 w_{m_2}. \)

This phenomenon is a type of peeling, however different from the Lorenz type bifurcation. In the Lorenz type, \( x_2 \) and \( x_3 \) peel of \( x_1 \). In the present system \( x_1 \) goes through certain bifurcation, and independent and away from \( x_1 \), the second generating singular point \( x_2 \) comes to existence in a similar fashion to the case in the system (2), Reference [6]. Global stability theorem must be satisfied for either \( x_1 \) or \( x_2 \), separately.

The stability properties of these two generating points are such that
\[
x_1: \quad s_1 < 0, \quad s_2 < 0, \quad s_3 > 0;
\]
\[
2 w_{m_1} + 1 w_{m_2},
\]
\[
x_2, 3: \quad s_1 < 0, \quad \text{Re} \ s_2 > 0, \quad \text{Re} \ s_3 > 0;
\]
\[
1 w_{m_1} + 2 w_{m_2}.
\]

However, separate from each other, these two points must obey the global stability theorem individually, not cooperatively, see [4]. This implies \( 1 w_{m_1} \) of \( x_1 \) and \( 2 w_{m_2} \) of \( x_2, 3 \) must be balanced separately. For the point \( x_2, 3 \) there are two possibilities, Fig. 3, these are the cases of a) hyperbolic neighborhood and b) elliptic neighborhood. In the case of (a) \( 2 w_{m_2} \) must be absorbed by two finite points resulting in a stable limit cycle \( L_2 \). In the case (b) \( 1 w_{m_1} \) is absorbed by \( 1 w_{m_1} \) thus only one \( w_{m_2} \) must be absorbed by a stable exploded point \( E_2 \).

Therefore,

Conclusion 2: The generating point \( x_1 \) is surrounded by a zero \( f \)-dimensional object, a stable exploded point \( E_1 \). The second generating singular point \( x_2, 3 \) is surrounded by another zero \( f \)-dimensional object, a stable exploded point \( E_2 \) (elliptic case) or a one \( f \)-dimensional object, a stable limit cycle \( L_2 \).

Thus the stable system bifurcates to yield \( x_1, E_1; \quad x_2, 3, E_2 \) (or \( L_2 \)). Three systems compared exhibit
the following topological variations initially:

Lorenz (1) \( x_1 \rightarrow \{x_1, x_2, x_3\} \)
System (2) None \( \rightarrow \{x_2, x_3\} \),
System (3) \( x_1 \rightarrow \{x_1, E_1 + x_2, x_3, E_2 \quad \text{(or } L_2)\}\).

The bifurcation of the second generating point of the system (3), while \( x_1, E_1 \) and \( E_2 \) (or \( L_2 \)) remain unaltered, takes place when \( B \) passes the \( B_2^* \) value. This is summarized below:

\[
\begin{align*}
x_1: & \quad s_1 < 0, \ s_2 < 0, \ s_3 > 0, \\
E_1: & \quad \text{a stable exploded point (dim } = 0), \\
E_2: & \quad \text{a stable exploded point (dim } = 0), \\
& \quad \text{(or stable limit cycle (dim } = 1)), \\
x_2 \rightarrow x_2: & \quad s_1 < 0, \ Re s_2 > 0, \ Re s_3 > 0; \\
& \quad 1w_{m1} + 2w_{m2}, \\
x_3: & \quad s_1 < 0, \ Re s_2 > 0, \ Re s_3 > 0; \\
& \quad 1w_{m1} + 2w_{m2}, \\
L_3: & \quad \text{a stable limit cycle (dim } = 1). \\
\end{align*}
\]

**Conclusion 3:** \( x_2 \) and \( x_3 \) cooperatively result in \( 2w_{m1} + 4w_{m2} \rightarrow 2w_{m2} \). Thus, a limit cycle \( L_3 \) a one dimensional object surrounds \( x_2 \) and \( x_3 \).

Therefore the following three cases are possible:

1. \( x_1 \),
2. \( \{x_1, E_1\} + \{x_2, x_3, E_2 \quad \text{(or } L_2)\}\),
3. \( \{x_1, E_1\} + \{x_2, x_3, E_2 \quad \text{(or } L_2), L_3\}\).

In these cases the Poincaré conditions are satisfied, [3]. Moreover, the global stability theorem [7] is satisfied throughout the alterations. The limit cycle \( L_3 \) may be similar to the limit cycle in system (2), thus it may be a limit bundle, [6].

We should also notice that if one looks at the stability property of the three singular points without making the distinction made above, one finds the collective stability characteristics of \( x_1, x_2 \) and \( x_3 \) as

\[
\begin{align*}
x_1: & \quad 2w_{m1} + 1w_{m2} , \\
x_2: & \quad 1w_{m1} + 2w_{m2}, \\
x_3: & \quad 1w_{m1} + 2w_{m2}, \\
& \quad 4w_{m1} + 5w_{m2} \rightarrow 1w_{m2}. \\
\end{align*}
\]

Thus the object which we identified as divided into parts \( E_1, E_2 \) (or \( L_2 \)) and \( L_3 \) are embedded into the exploded point \( E \),

\[
E = E_1 \oplus E_2 \quad \text{(or } L_2) \oplus L_3. 
\]

This would imply that in order to offset \( L_3 \) (and \( L_2 \)), there must be unstable limit cycle(s) \( L^u \) embedded in \( E \), however their existence can only be suggested indirectly as argued above. The limit cycle(s) as well as the exploded points are observed by simulation in [5], particularly for the special case of \( d = 0 \), i.e., \( B = 0 \), see Table 1. One of the exploded points is illustrated in Figure 4.

**Fig. 4.** An exploded point of the system (3) with the parameter values, \( a = 0.1, b = 0.07, c = 0.38, d = 0.0 \).

In summary we have shown that following the second bifurcation the three systems considered yield the cases below:

The system (1): Cooperative peeling of multiple singular points, has a unique exploded point [1, 3 and 4].

The system (2): Degenerate peeling of a single singular point, has a unique limit bundle [6].

The system (3): Independent peeling of multiple singular points, has multiple exploded points and limit bundles.