The V-V Sector of the Lee Model

Herbert Rainer Petry and Mikolaj Sawicki*
Institut für Theoretische Kernphysik der Universität Bonn
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The Schrödinger-equation in the V-V sector of the Lee model is investigated. We obtain a three-body Lippmann-Schwinger equation, with a new type of disconnectedness, and construct the resolvent operator by means of a modified Faddeev technique.

1. Introduction

The Lee-model [1] provides us with a soluble field-theory, which allows a detailed discussion of renormalization effects. It exists in a variety of forms, the static model [1], the quasi-relativistic model [2] and the “Galilee” model [3, 4] which have in common the peculiar interaction $V \leftrightarrow N \theta$ between two kinds of fermions $V$, $N$ and a boson $\theta$, and differ otherwise in the specific choice of the free single-particle energies and the interaction form factor. In this paper we concentrate mainly on the so-called $V$-$V$-sector which is the most difficult one considered so far [5]. As we will show in Sect. 2, it requires the solution of a Faddeev-type equation with renormalized energies. We assume in this paper that the coupling constant renormalization is always finite, which can be achieved, e.g., in the Galilee model or for specific form factors, but we allow for infinite energy renormalization.

Section II reviews the $V$ and the $V\theta$-sector. In Sect. III we study the $V$-$V$-sector, eliminate the unrenormalized quantities by renormalized ones and show, how the solutions of this sector can be obtained by the techniques developed by Faddeev [6].

2. The Lee-Model and its First Two Sectors

The Hamiltonian of the Lee-model is given by

$$H = H_0 + H_I,$$

where

$$H_0 = \int \frac{d^3p}{(2\pi)^3} [\omega_X(p) V^+(p)V(p) + \omega_Y(p) N^+(p)N(p) + \varepsilon(p)[a^+(p)a(p)]],$$

and

$$H_I = \lambda_0 \left[ \int \frac{d^3p}{(2\pi)^3} \frac{d^3q}{(2\pi)^3} \frac{1}{2} \left[ V^+(p)N(p-q)a(q)h(q) + N^+(p-q)a^+(p)V(p)h(q) \right] \right]$$

with the form factor function

$$h : \mathbb{R}^3 \to \mathbb{C}.$$
of the state vector in this sector is
\[ \psi_V = [\int d^3k \varphi(k) V^+(k) + \int d^3l d^3m \beta(l, m) N^+(l) a^+(m)] |0\rangle. \]

We require that the state \( \psi_V \) is an eigenstate of the total Hamiltonian \( H \) with eigenvalue \( E \), i.e.
\[ (E - H_0) \psi_V = H_I |\psi_V\rangle. \]

Comparing the coefficients of both components of the state vector \( \psi_V \) we obtain the equations
\[ (E - \omega_V(k)) \beta(k, q, q) = \lambda_0 \beta(k, q, m) \beta(k - q, m), \]
\[ \beta(l, m) = \lambda_0 h(m)[E - \omega_N(l) - \epsilon(m)]^{-1} \beta(l + m). \]

They imply that
\[ R(E, k) \beta(k, q, q) = 0 \]
with
\[ R(E, k) = E - \omega_V(k) = \lambda_0^2 \int d^3m h(q) \cdot [E - \omega_N(k - m) - \epsilon(m)]^{-1} \]
and \( h(q) = |h(q)|^2 \).

A nontrivial solution of Eq. (0) exists if and only if \( R(E, k) \) vanishes for some value of \( E \) denoted by \( \omega_V(k) \).

The quantity \( \omega_V(k) \) has the meaning of a renormalized, i.e. physical, energy of the \( V \)-particle.

Evidently
\[ \omega_V(k) = \omega_N(k - m) - \epsilon(m) \neq 0 \]
must hold (for all \( k, m \)), if the last formula has any meaning. If this condition is not satisfied, the \( V \)-particle will decay, a case which we exclude. We also exclude the possibility that the equation
\[ R(E, k) = 0 \]
has other solutions than \( \omega_V(k) \).

Since \( R(\omega_V(k), k) = 0 \), we redefine
\[ R(E, k) = R(E, k) - R(\omega_V(k), k) \]
which yields
\[ \psi_{V \theta} = [\int d^3k d^3m \varphi(k, m) V^+(k) a^+(m)] |0\rangle + \int d^3k d^3m_1 d^3m_2 \beta(k, m_1, m_2) N^+(k) a^+(m_1) a^+(m_2) |0\rangle. \]

Bose-Einstein statistics requires that
\[ \beta(k, m_1, m_2) = \beta(k, m_2, m_1). \]

If \( \psi_{V \theta} \) is an eigenstate of energy, it must hold that:
\[ (E - H_0) \psi_{V \theta} = H_I \psi_{V \theta} \]
which yields
\[ \psi_{V \theta} = [\int d^3k \varphi(k, m) V^+(k) a^+(m)] |0\rangle + \int d^3k d^3m_1 d^3m_2 \beta(k, m_1, m_2) N^+(k) a^+(m_1) a^+(m_2) |0\rangle. \]
Eliminating $\beta$ by means of the last formula leads to the equation

$$R(E - \epsilon(m), k) x(k, m) = \lambda_0^2 h(m) \int d^3q \, \bar{h}(q)$$

$$\cdot \left[ (E - \epsilon(m) - \omega_N(k - q) - \epsilon(q))^{-1} \right. x(k - q + m, q).$$

Using formula (2), which redefines $R$ with help of the renormalized energy $\omega^R$ and introducing the function $\bar{z}(k, l)$ by

$$\bar{z}(k, l) = z(k, l)/h(l)$$

we find immediately

$$(E - \omega^R_R(k) - \epsilon(m))$$

$$\cdot Z(\omega^R_R(k), E - \epsilon(m), k) \bar{z}(k, m)$$

$$= \lambda_0^2 \int d^3q \, q(q) \left[ (E - \epsilon(m) - \omega_N(k - q) - \epsilon(q))^{-1} \right.$$  

$$\cdot \bar{z}(k - q + m, q).$$

This equation shows that the $V^0$-sector is indeed fully described by renormalized quantities.

3. The $V^V$-Sector

3.1. Renormalization

The $V^V$-sector is characterized by the values $Q_1 = 2$ and $Q_2 = 2$; the general form of a state in this sector is

$$\varphi_{VV} = \int d^3k_1 d^3k_2 \, \varphi(k_1, k_2) V^+(k_1) V^+(k_2) |0>$$

$$+ \int d^3k d^3m \, \beta(k, l, m) V^+(k) N^+(l) a^+(m) |0>$$

$$+ \int d^3l_1 d^3l_2 d^3m_1 d^3m_2 \, \gamma(l_1, l_2, m_1, m_2) \gamma(l_1, l_2, m_1, m_2) N^+(l_1) N^+(l_2) a^+(m_1) a^+(m_2) |0>$$

and the correct statistics requires that

$$\alpha(k_1, k_2) = -\alpha(k_2, k_1), \quad \gamma(l_1, l_2, m_1, m_2) = -\gamma(l_2, l_1, m_1, m_2) = \gamma(l_1, l_2, m_1, m_2).$$

If $\varphi_{VV}$ is an eigenfunction of the Hamiltonian $H$ with eigenvalue $E$, then

$$(E - H_0) \varphi_{VV} = H_1 \varphi_{VV},$$

which implies that

$$[E - \omega^R_R(k) - \omega_N(k) - \epsilon(m)] \beta(k, l, m) = A_{k, k_1} \lambda_0 \int d^3q \, \bar{h}(q) \beta(k_1, k_2 - q, q),$$

$$[E - \omega^R_R(k) - \omega_N(l) - \epsilon(m)] \beta(k, l, m) = 2\lambda_0 h(m) \alpha(k, l + m) + 4 \lambda_0 \int d^3q \, \bar{h}(q) \gamma(k - q, l, q, m)$$

and

$$[E - \omega_N(l_1) - \omega_N(l_2) - \epsilon(m_1) - \epsilon(m_2)] \gamma(l_1, l_2, m_1, m_2) = -A_{l_1, l_2} S_{m_1 m_2} \lambda_0 h(m_2) \beta(l_2 + m_2, l_1, m_1)$$

where $A_{l_1, l_2}$ and $S_{m_1 m_2}$ are the antisymmetrisation and symmetrisation operators in fermion and boson momenta, respectively,

$$A_{l_1, l_2} f(l_1, l_2) = \frac{1}{2} (f(l_1, l_2) - f(l_2, l_1))$$

and $S_{m_1 m_2}$ has been defined in the last section. We define $\tilde{\beta}$ by

$$\tilde{\beta}(k, l, m) = \beta(k, l, m)/h(m)$$

and eliminate $\gamma$. The result is given by the equations:

$$(E - \omega_N^R(k) - \omega_N(l) - \epsilon(m)) \tilde{\beta}(k, l, m)$$

$$= 2\lambda_0 \alpha(k, l + m) + \lambda_0^2 \int d^3q \, q(q) \left[ (E - \omega_N(l) - \epsilon(m) - \omega_N(k - q) - \epsilon(q))^{-1} \right.$$  

$$\cdot \left[ \tilde{\beta}(k - q + m, l, q) - \tilde{\beta}(l + m, k - q, q) + \tilde{\beta}(k, l, m) - \tilde{\beta}(l + q, k - q, m) \right]$$

and

$$(E - \omega_N^R(k_1) - \omega_N^R(k_2)) \alpha(k_1, k_2) = \frac{1}{2} \lambda_0 \int d^3q \, q(q) \left[ \bar{\beta}(k_1, k_2 - q, q) - \bar{\beta}_2(k_2, k_1 - q, q) \right].$$

We observe that the third term under integral sign in (3) provides us, in an analogy to the $V^0$-sector, with the off-shell energy renormalization of the $V$-particle energy $\omega^R_R(k)$; proceeding as in the previous section we rewrite Eq. (3) in the form

$$[E - \omega^R_R(k) - \omega_N(l) - \epsilon(m)] Z(E - \omega_N(l) - \epsilon(m), \omega^R_R(k), k) \tilde{\beta}(k, l, m)$$

$$= 2\lambda_0 \alpha(k, l + m) + \lambda_0^2 \int d^3q \, q(q) \left[ (E - \omega_N(l) - \epsilon(m) - \omega_N(k - q) - \epsilon(q))^{-1} \right.$$  

$$\cdot \left[ - \tilde{\beta}(l + q, k - q, m) + \tilde{\beta}(k - q + m, l, q) - \tilde{\beta}(l + m, k - q, q) \right]$$

and

$$[E - \omega^R_R(k_1) - \omega^R_R(k_2)] \alpha(k_1, k_2) = \frac{1}{2} \lambda_0 \int d^3q \, q(q) \left[ \tilde{\beta}(k_1, k_2 - q, q) - \tilde{\beta}_2(k_2, k_1 - q, q) \right].$$
where \( Z(E - \omega_N(l) - \epsilon(m), \omega_-^R(k), k) \) is again the half-off-shell value of \( Z(E, E', k) \) defined in the last section. Equation (5) is then expressed by renormalized quantities only. We try to do this for (4), too.

To this end we first rewrite the Eq. (5) in the form of a Lippmann-Schwinger equation; (hereafter we shall use the symbols \( k, l \) and \( m \) to describe the momenta of particles \( V, N \) and \( \theta \), respectively):

\[
\int d^3k' d^3l' d^3m' \langle k, l, m \mid G^R_{03}(E)^{-1} \mid k', l', m' \rangle \tilde{\beta}(k', l', m') = 2 \lambda_0 \mathcal{A}(k, l + m) + \int d^3k' d^3l' d^3m' \langle k, l, m \mid V(E) \mid k', l', m' \rangle \tilde{\beta}(k', l', m'),
\]

where

\[
\langle k, l, m \mid G^R_{03}(E)^{-1} \mid k', l', m' \rangle = \delta(P - P') \delta(l - l') \delta(m - m') [E - \omega_-^R(k) - \omega_N(l) - \epsilon(m)] \cdot Z(E - \omega_N(l) - \epsilon(m), \omega_-^R(k), k)
\]

and \( P, P' \) are the total momenta of the system:
\[ P = k + l + m, \quad P' = k' + l' + m'. \]

The potential \( V \) has the form
\[ V(z) = V_1(z) + V_2(z) + V_x(z) \]
with

\[
\langle k, l, m \mid V_1(E) \mid k', l', m' \rangle = -\delta(P - P') \delta(m - m') \lambda_0^2 \mathcal{Q}(k - l') [E - \omega_N(l) - \epsilon(m) - \omega_N(l') - \epsilon(k - l')]^{-1},
\]

\[
\langle k, l, m \mid V_2(E) \mid k', l', m' \rangle = +\delta(P - P') \delta(l - l') \lambda_0^2 \mathcal{Q}(m') [E - \omega_N(l) - \epsilon(m) - \omega_N(k - m') - \epsilon(m')]^{-1},
\]

and

\[
\langle k, l, m \mid V_x(E) \mid k', l', m' \rangle = -\delta(P - P') \delta(k - l' - m') \lambda_0^2 \mathcal{Q}(m') [E - \omega_N(l) - \epsilon(m) - \omega_N(k - m') - \epsilon(m')]^{-1}.
\]

The detailed discussion of the potentials is given in the next section. We define now the total Green operator \( G \) by

\[
G(E) = [G^R_{03}(E)^{-1} - V(E)]^{-1},
\]

and rewrite the Eq. (6) in the form
\[
\tilde{\beta}(k, l, m) = \int d^3k' d^3l' d^3m' \langle k, l, m \mid G(E) \mid k', l', m' \rangle 2 \lambda_0 \mathcal{A}(k', l' + m').
\]

Substituting the last equation into (4) we get

\[
[E - \omega_-^R(k_1) - \omega_-^R(k_2)] \mathcal{A}(k_1, k_2) = \lambda_0^2 \int d^3q \mathcal{Q}(q) \int d^3k' d^3l' d^3m' \langle k_1, k_2 - q, q \mid G(E) \mid k', l', m' \rangle \mathcal{A}(k', l' + m') - (k_1 \leftrightarrow k_2). \tag{7}
\]

We introduce now the \( T \)-operator for the \( VN\theta \)-system:
\[
T(z) = V(z) + V(z) G(z) V(z). \tag{8}
\]

The following identities hold
\[
G(z) = G^R_{03}(z) + G^R_{03}(z) T(z) G^R_{03}(z) \tag{9}
\]
and
\[
T(z) = V(z) + V(z) G^R_{03}(z) T(z). \tag{10}
\]

We insert now (9) into Eq. (7) and observe that the first term of Eq. (9) provides us with the renormalization of the energies of the \( V \)-particles and that we can write

\[
[E - \omega_-^R(k_1) - \omega_-^R(k_2) + AE(k_1, k_2)] \mathcal{A}(k_1, k_2) = \lambda_0^2 \int d^3q \mathcal{Q}(q) \int d^3k' d^3l' d^3m' \langle k_1, k_2 - q, q \mid G^R_{03}(E) \cdot T(E) \cdot G^R_{03}(E) \mid k', l', m' \rangle \mathcal{A}(k', l' + m') - (k_1 \leftrightarrow k_2),
\]
where
\[ AE(k_1, k_2) = \lambda_0^2 \int \! d^3m \, \rho(m) \{ [\omega_N(k_2 - m) - \omega_N(k_2 - m) - \varepsilon(m)]^{-1} \]
\[ - \{ [E - \omega_N(k_2 - m) - \omega_N(k_2 - m) - \varepsilon(m)]^{-1} \cdot Z (E - \omega_N(k_2 - m) - \varepsilon(m), \omega_R(k_1), k_1) \} + (k_1 \leftrightarrow k_2). \]

is a finite quantity. This achieves our goal to express (7) in terms of renormalized energies.

### 3.2. The Faddeev equations

We are now coming to the construction of the three-body operator, defined by Eq. (8); to this end we consider the potentials \( V_1, V_2 \) and \( V_X \), defined in the last section. The potentials \( V_1 \) and \( V_2 \) correspond to the two-body energy-dependent interactions \( V - N \) and \( V - \theta \) imbedded in a three-body space, as is exhibited by the presence of delta functions conserving the momentum of spectator particles \( \theta \) and \( N \), respectively. Therefore, they give rise to the usual disconnected graphs in the three-body space, shown in Figures 1a, 1b. (The overall delta function, common for all three potentials represents, of course, conservation of the total momentum.) In contrast to \( V_1 \) and \( V_2 \), the interaction \( V_X \) changes all individual momenta of the particles, but due to the additional delta function gives still rise to the disconnected diagram 1c, yielding in this way a new type of a quasi-three-body force specific for the Lee-model.

It is interesting to note that the iterated product \( V_X G_{03}^R V_X \) generates the conservation of momentum of the \( V \)-particle as it is shown in Fig. 2, and, therefore, corresponds to the two-body \( N\theta \)-interaction originally not present in the potential of our Lippmann-Schwinger equation.

\[
\begin{array}{ccc}
 k & v & k' \\
 m & i & m' \\
\end{array} = \begin{array}{ccc}
 k & v & k' \\
 m & i & m' \\
\end{array},
\]

\[
\begin{array}{ccc}
 k & v & k' \\
 m & i & m' \\
\end{array} = \begin{array}{ccc}
 k & v & k' \\
 m & i & m' \\
\end{array},
\]

\[
\begin{array}{ccc}
 k & v & k' \\
 m & i & m' \\
\end{array} = \begin{array}{ccc}
 k & v & k' \\
 m & i & m' \\
\end{array},
\]

Fig. 1. Graphical representation of the disconnectedness in the potentials \( V_1, V_2 \) and \( V_X \).

Fig. 2. The disconnectedness generated by the product \( V_X G_{03}^R V_X \).

In order to construct \( T \) we have to sum up all diagrams exhibiting the same type of disconnectedness. To this end we consider possible types of iterations in Equation (10). Following Stingl [7] we introduce a symbolic notation \( \delta_N, \delta_\theta, \delta_x \) and \( \delta_y \) for the delta functions in \( V_1, V_2, V_X \) and \( V_X G_{03}^R V_X \) respectively and write, for instance "\( \delta_x \delta_x \rightarrow \delta_y \)”, i.e. two terms with momentum \( \delta \)-functions \( \delta_x \) and \( \delta_y \) for the delta functions in \( V_1, V_2, V_X \) and \( V_X G_{03}^R V_X \), sandwiched in between, give after operator multiplication a term with a \( \delta \)-function \( \delta_y \). By inspection of all possible diagrams we find the following "disconnectedness multiplication table" (see Fig. 3) where the empty box means that the corresponding "disconnectedness multiplication" leads to a fully connected graph.

Now it is quite easy to sum up all graphs with a given type of disconnectedness: we define
\[
t_1 = V_1 + V_1 G_{03}^R V_1 + V_1 G_{03}^R V_1 G_{03}^R V_1 + \cdots,
\]
\[
t_2 = V_2 + V_2 G_{03}^R V_2 + V_2 G_{03}^R V_2 G_{03}^R V_2 + \cdots,
\]
\[
t_3 = V_X + V_X G_{03}^R V_X G_{03}^R V_X + \cdots,
\]
\[
t_x = V_X G_{03}^R V_X + V_X G_{03}^R V_X G_{03}^R V_X + \cdots,
\]

and observe that the following integral equations hold:
\[
t_1 = \int V_1 + V_1 G_{03}^R t_1,
\]
\[
t_2 = \int V_2 + V_2 G_{03}^R t_2,
\]
\[
\begin{pmatrix}
 t_x \\
 t_y \\
\end{pmatrix} = \begin{pmatrix}
 V_x \\
 0 \\
 V_x \\
 0 \\
\end{pmatrix} G_{03}^R \begin{pmatrix}
 t_x \\
 t_y \\
\end{pmatrix}.
\]

We introduce now the channel Green operators \( G_x \) for the three potentials \( V_1, V_2 \) and \( V_X \) contributing to the full Green operator \( G \),
\[
G_x(z) = (G_{03}^R(z))^{-1} - V_x(z))^{-1}, \quad x = 1, 2, x,
\]

and observe that the following relations hold:
\[
V_1 G_i = t_i G_{03}^R, \quad i = 1, 2, \quad (12)
\]
\[
V_X G_x = (t_x + t_y) G_{03}^R. \quad (13)
\]
Now we repeat Faddeev's procedure and split the total operator $T$ into three parts
$$T = T_1 + T_2 + T_3$$
where
$$T_1 = V_1 + V_1 GV,$$
$$T_2 = V_2 + V_2 GV,$$
$$T_3 = V_x + V_x GV.$$ Using the resolvent identity
$$G = G_x + G_x V_x G, \quad x = 1, 2, x,$$
where
$$r_x = V - V_x$$
we obtain with help of (12) for $i = 1, 2$:
$$T_i = t_i + t_i G_x \sum_{x \neq i} T_x ,$$
and, using Eq. (13)
$$T_3 = t_3 + t_3 G_x (T_1 + T_2),$$
where we defined
$$t_3 = t_x + t_y.$$ We rewrite the last equations in the form
$$T_i = t_i + t_i G_x \sum_{i \neq j} T_j , \quad i = 1, 2, 3, \quad (14)$$
and observe that the kernel of our Faddeev equations (14) becomes connected after a single iteration so that (14) could be solved in the standard way. The solutions $T_i$ have the form
$$T_i = t_i + W_i ,$$
where $W_i$ is the fully connected part of $T_i$. Inserting the last formula in Eq. (11) yields
$$\left[ E - cov^R (k_1) - cov^R (k_2) + \Delta E (k_1, k_2) + \Delta E_3 (k_1, k_2) \right] \chi (k_1, k_2)
= \lambda_0 \int d^3 q \left( q \right) d^3 k' d^3 l' d^3 m' \left< k_1, k_2 - q, q \right| G^R (E) \left[ T_1 (E) + T_2 (E) + W_3 (E) \right] \cdot G^R (E) \left| k', l', m' \right> \chi (k, l, m) - (k_1 \leftrightarrow k_2). \quad (15)$$
The additional finite energy shift $\Delta E_3 (k_1, k_2)$ is calculated in Appendix A.

The homogeneous Eq. (15) corresponds to an effective two-body problem. Its solutions can be calculated, at least numerically, by standard methods. In the next step Eq. (6) is solved for $\tilde{\beta} (k, l, m)$. Knowing $\beta (k, l, m)$ we obtain immediately the function $\gamma (l_1, l_2, m_1, m_2)$.

### Appendix A

We demonstrate now that the term $G_x G_x$ produces an additional energy shift $\Delta E_3 (k_1, k_2)$. To this end we calculate
$$I = \int d^3 q \left( q \right) d^3 k' d^3 l' d^3 m' d^3 k'' d^3 l'' d^3 m'' \left< k_1, k_2 - q, q \right| G^R \left| k', l', m' \right> \left< k'', l'', m'' \right| T_x + t_x \left| k'', l'', m'' \right>$$

$$\left< k'', l'', m'' \right| G^R \left| k', l', m' \right> \chi (k', l' + m') - (k_1 \leftrightarrow k_2). \quad (A1)$$

We introduce the following notation
$$\left< k, l, m \right| G^R \left| k', l', m' \right> = \delta (k - k') \delta (l - l') \delta (m - m') g_0 (E, k', l', m'),$$
$$\left< k, l, m \right| t_x (E) \left| k', l', m' \right> = \delta (k + l + m - k' - l' - m') \delta (k - l' - m') \tau_x (E, k, l, m, m'),$$
$$\left< k, l, m \right| t_y (E) \left| k', l', m' \right> = \delta (k + l + m - k' - l' - m') \delta (k - k') \tau_y (E, k, l, m, m'). \quad (A2)$$

Inserting (A2) into (A1) and performing the integration over all delta functions we obtain
$$I = - \Delta E_3 (k_1, k_2) \chi (k_1, k_2)$$

with
$$\Delta E_3 (k_1, k_2) = \int d^3 q \left( q \right) d^3 l' g_0 (E, k_1, k_2 - q, q) \tau_x (E, k_1, k_2 - q, q, k_1 - l') g_0 (E, k_2, l', k_1 - l')$$

$$- \int d^3 q \left( q \right) d^3 l' g_0 (E, k_1, k_2 - q, q) \tau_y (E, k_1, k_2 - q, q, k_2 - l')$$

$$\cdot g_0 (E, k_1, k_2 - l) + (k_1 \leftrightarrow k_2). \quad (16)$$