Radiationless Internal Vibrations of Extended Charged Particles in a Constant Magnetic Field

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(Z. Naturforsch. 32 a, 903—913 [1977] ; received July 9, 1977)

The extended particle theory recently proposed is further investigated. Especially, it is shown for the force-free case that no runaway solutions exist and that the particle must ultimately assume constant velocity. For a certain structure of the particle, internal radiationless vibrations are found and discussed extensively for the case, where the particle moves in a constant homogenous magnetic field. If the magnetic field strength assumes a certain discrete set of values, the motion terminates in that radiationless internal rotation; in all other cases the particle spirals inwards under emission of radiation until it comes to rest.

I. Introduction and Survey of Results

In two recent papers, Moniz and Sharp have shown within the range of validity of non-relativistic quantum electrodynamics, that the equation of motion of the radiating electron is essentially of non-local character, even in the point-particle limit. From the point of view of relativistic quantum electrodynamics, this seems to be quite plausible, because the original point-like particle creates electron-positron pairs in its immediate vicinity (vacuum polarization); and since the positrons are attracted and the virtual electrons are repelled by the original point charge, there must arise a charge “atmosphere” (as Blokhinsev calls it) with a certain distribution of the charge around the point particle. The linear dimensions of this polarization cloud are in the order of magnitude of the Compton wavelength $\lambda_c (\approx \hbar/2 mc)$, which just enters the equation of Moniz and Sharp as a non-locality parameter.

Having in mind this concretization of the semi-classically radiating electron, one can try to find the classical equation of motion for such an extended particle, the charge distribution of which might now be described from a purely classical point of view, though its existence is originally the consequence of a quantum effect. The only difference with respect to a purely classical theory of the radiating electron would then consist in the modified dimension of the particle: the dominant length parameter is the Compton wavelength $\lambda_c (\approx \hbar/2 mc)$ instead of the classical electron radius $r_0 (\approx Z^2/mc^2)$. However, since the numerical value of this length parameter (called $\lambda_s$ hereafter) is itself not derivable from such a classical model, it must not be specified a priori but it can be fitted later in order to meet with experimental data or/and with a higher level theory. Such a higher level theory might be relativistic quantum electrodynamics or (e.g.) the chronon theory of Caldirola.

In a recent paper, we have presented a covariant framework for arbitrarily extended, classical, charged particles being characterized by certain time-like structure functions. In the present paper, we want to study some specific effects occurring in connection with such extended particles: these are the internal self-oscillations, which are excited and sustained by external forces of appropriate frequency.

If one considers namely the elementary length interval $\lambda_s$ as a characteristic time parameter of the particle, one would expect internal damped oscillations of frequency $\omega_s^{-1}$, which is the inverse propagation time of a light signal running through the particle; and whenever the external force changes appreciably over proper time intervals of magnitude $\lambda_s$, those internal oscillations could be excited in principle. The result would be a sort of “resonance” between the internal modes and the external force such that certain of the modes become stationary and radiationless, and the particle would be left in a stationary excited state in the given force field.

It is exactly this phenomenon, which we intend to demonstrate explicitly in this paper by means of the recently proposed non-local equation of motion for the classically radiating electron

$$m_{\text{mech}} c^2 \ddot{u} + m_{\text{el}} c^2 \left( \frac{\dot{u}}{\lambda_s} - \left( \dot{u} \cdot \frac{\dot{u}}{\lambda_s} \right) \dot{u} \right) = K c^2 \cdot (I, 1)$$

In contrast to the previously studied one-dimensional case, we turn now to a two-dimensional...
problem and choose for the external force a constant uniform magnetic field

\[ K^i = Z F^{i\mu} u_{\mu} \quad (I, 2) \]

with

\[ -F^{21} = F^{12} = B = \text{const} \quad (I, 3) \]

and all other components of the field strength tensor vanishing.

Under these conditions, we find that for the discrete set of magnetic field strengths

\[ B_n = | Z |^{-1} m_{\text{mech}} c^2 \omega_n \]

with the cyclotron frequencies \( \omega_n = n \cdot 2 \pi / \Delta s \) there exist circular radiationless motions, where the radius of the circles is dependent upon the initial condition but never exceeds the extension parameter \( \Delta s \). Thus, it is clear, that those motions are internal rotations of the extended particle.

These results also apply to the non-relativistic limit, where the spectral decomposition of an arbitrary motion with respect to the internal decaying modes is easily feasible by means of the Laplace transform method. One can demonstrate here explicitly, how the various damping constants of the internal modes decrease (with increasing magnetic field strength \( B \)) down to zero for the special values \( B_n \) from above. We present numerical results of how the stationary circles are approached in the course of the motion starting with a constant impact velocity as initial condition.

The longitudinal motion (parallel to the magnetic lines of force) is also studied, and it is found that this longitudinal motion is rapidly damped down to constant velocity, as well for arbitrary values of \( B \), where the particle spirals inwards until it comes to rest in the transverse plane as also for the above “resonant” values \( B_n \) of the magnetic field, where the final state of motion is such an internal rotation. Hence, one might say from a macroscopic point of view that the longitudinal velocity is “approximately” a constant of the motion.

II. Existence of Radiationless Motions

First, we want to show that the equation of motion (I, 1) is the only one of the general class

\[ m_{\text{mech}} c^2 \dot{u}^i + m_{\text{el}} c^2 \left( \dot{Q}^i - (\dot{Q} u) u^i \right) = K^i \quad (II, 1) \]

where

\[ \dot{Q}^i = \int_0^{\Delta s} f_{(s)} u^i(s-\sigma) \, d\sigma / \Delta s , \quad (II, 2) \]

which admits stationary radiationless motions for a force \( K^2 \), which itself does not transfer energy to the particle [e.g. magnetic force (I; 2, 3)]. To this end, assume that there be a periodic motion of proper time period \( \Delta s \)

\[ u^i(s) = u^i(s + \Delta s) , \quad \forall s \quad (II, 3) \]

so that an integration of (II, 1) yields on account of

\[ \int_s^{s+\Delta s} K^0(s) \, ds' = 0 , \quad \forall s \quad (II, 4) \]

the following result \( (z_0 \equiv c \, t; \, dz^i / ds \equiv u^i) \)

\[ m_{\text{mech}} c^2 [u^0]_s^{s+\Delta s} + m_{\text{el}} \int_0^{\Delta s} f_{(s)} [u^0(s-\sigma)]_s^{s+\Delta s} \]

\[ = m_{\text{el}} c^2 \int_{t_{(s)}}^{t_{(s+\Delta s)}} (\dot{Q} u) \, dt . \quad (II, 5) \]

Because of the periodicity condition (II, 3), the first two integrals on the left of (II, 5) vanish and we are led to the requirement

\[ \int_{t_{(s)}}^{t_{(s+\Delta s)}} (\dot{Q} u) \, dt = 0 , \quad \forall s . \quad (II, 6) \]

But the invariant radiation rate \( \Re \)

\[ \Re = -m_{\text{el}} c (\dot{Q} u) \]

\[ = m_{\text{el}} c \left\{ \int_0^{\Delta s} f_{(s)} [u^i(s) u^i(s-\sigma) - 1] \right. \]

\[ - \left. \int_0^{\Delta s} f_{(s)} [u^i(s) u^i(s-\sigma) - 1] \, d\sigma / \Delta s \right\} \quad (II, 7) \]

is positive-semidefinite on account of the restrictions imposed on the structure function \( f_{(s)} \):

\[ f_{(s)} \geq 0 ; \quad f_{(s)} \equiv 0 \quad (II, 8) \]

and hence we must have \( \Re \equiv 0 \) along the trajectory. However, inspection of (II, 7) shows that the radiation rate can vanish only in two cases:

\[ a) \quad f_{(s)} (\sigma) = 0 \quad \forall \sigma \in [\sigma_1, \sigma_2] \subseteq [0, \Delta s] ; \quad \sigma_2 > \sigma_1 . \quad (II, 9) \]

In this case, the radiation rate \( \Re \) vanishes only if \( u^i = \text{const.} (s) \) and the only possible stationary motion in the force field under consideration is uniform motion [e.g. parallel to the magnetic lines of force in example (I, 3)].

\[ \beta) \quad f_{(s)} (\sigma) \equiv 0 ; \quad f_{(s)} \equiv 1 ; \quad u^i(s - \Delta s / n) = u^i(s) \]

\[ n = 1, 2, 3, \ldots . \quad (II, 10) \]

For this special choice of \( f_{(s)} \), the general equation of motion (II, 1) reduces to (I, 1); and we see that the proper time period \( \Delta s \) must be equal to the
fundamental time interval $\Delta s/n$. For this reason, the electromagnetic four-momentum $P_{el}^a$

$P_{el}^a := m_{el} c Q^a = m_{el} c \Delta s^{-1} (z^a(s) - z^a(s - \Delta s)) \quad (I, 10)$

vanishes also ($P_{el}^a \equiv 0, \forall s$) and Eq. (I, 1) assumes the well-known, non-radiative form

$m_{\text{mech}} c^2 \dot{u}^2 = K^3 \quad (II, 12)$

which is known to admit circular motions as solutions in a constant homogeneous magnetic field (see next sections).

We see that the electromagnetic part has disappeared from the equations, and the behaviour of the particle in such a “resonant” motion is determined only by its mechanical subsystem. Moreover, our considerations have been performed completely within the framework of Special Relativity so that one does not have to fear the spuriousness of the corresponding non-relativistic linearized calculations (see Sect. 6 G of Erber’s work 8).

An obvious generalization of the structure function (II, 10), which still leads to radiationless motions, is (e.g.)

$f'(\sigma) = -\kappa \delta(\sigma - \sigma_0) ; \quad f(\sigma) = \begin{cases} f_1 = \text{const} ; 0 < \sigma < \sigma_0 < \Delta s , \\ f_2 = \text{const} ; \sigma_0 < \sigma \leq \Delta s . \end{cases} \quad (II, 13)$

The three constants $\kappa, f_1, f_2$ are connected by the normalization condition 5

$\frac{\Delta s}{\int f(\sigma) \frac{d\sigma}{\Delta s}} = 1 \quad (II, 14)$

and the ‘charge normalization’ condition 5

$\int_0^\sigma f(\sigma) \frac{d\sigma}{\Delta s} = \frac{2}{3} Z^2 (m_{el} c^2)^{-1} = : \bar{\sigma} \quad (II, 15)$

through

$f_1 = (\Delta s + \sigma_0 - 2 \bar{\sigma}) \sigma_0^{-1} \quad (II, 16a)$

$f_2 = (2 \bar{\sigma} - \sigma_0) (\Delta s - \sigma_0)^{-1} \geq 0 \quad (II, 16b)$

Moreover

$f_1 - f_2 = \kappa = \Delta s (\Delta s - 2 \bar{\sigma}) \sigma_0^{-1} (\Delta s - \sigma_0)^{-1} > 0 \quad (II, 17)$

The radiation rate $\Re$ from (II, 7) then vanishes on account of the periodicity condition (assume $\sigma_0/\Delta s$ as a rational number):

$u^i(s) = u^i(s - n \sigma_0) = u^i(s - m \Delta s), m, n = 1, 2, 3, \ldots \quad (II, 18)$

and the electromagnetic four-momentum corresponding to (II, 2) becomes

$P_{el}^a = \Delta s^{-1} m_{el} c \{ f_1 [z^a(s) - z^a(s - \Delta s)] \quad (II, 19)$

$+ f_2 [z^a(s - \sigma_0) - z^a(s - \Delta s)] \}$.

Obviously, such a particle structure resembles a piecewise constant “charge distribution”.

Summarizing results, we have found that non-trivial radiationless stationary motions in external force fields, which do not supply energy to the particle during the stationary motion, are only possible in the case of piecewise constant structure functions $f(\sigma)$ in the class of Equations (II, 1). We proceed now to analyze these stationary states in more detail.

III. Radiationless Motions in the Relativistic Case

In this section, we restrict ourselves to the motion of the particle in a plane orthogonal to the direction of the constant homogeneous B-field. The equation of motion (I, 1) in the radiationless case reduces to (II, 12) together with the periodicity condition (II, 10). Since it is already known that the only solutions of (II, 12) for the special force field (I; 2, 3) are circles, we can at once insert the ansatz

$z^0(\sigma) = b \ast \sigma , \quad z^2(\sigma) = A \sin (\omega \sigma) , \quad z^3(\sigma) = 0 \quad (III, 1)$

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with
\[ u^2 u_1 = 1 \Rightarrow b^* = \pm \sqrt{1 + (A \omega)^2} \quad (III, 2) \]
into the reduced set of equations
\[
\begin{align*}
\text{mech} c^2 \dot{u}_1 &= Z B u^2, \\
\text{mech} c^2 \dot{u}_2 &= -Z B u^1,
\end{align*}
\tag{III, 3}
\]
to obtain
\[
\text{mech} c^2 \omega = -Z B = |Z| \cdot B. \quad (III, 4)
\]
The cyclotron frequency \( \omega_c \), if measured in proper time units, is fixed by the strength of the \( B \)-field alone, independent from the energy of the particle and the initial conditions.

But now we have to add the periodicity condition
\[ u^2(s) = u^2(s + \Delta s) \], which selects the frequency \( \omega \) as
\[ \omega_n \Delta s = n \cdot 2 \pi, \quad n = 1, 2, 3, \ldots \quad (III, 5) \]
and this, conversely, fixes the \( B \)-field as
\[ B_n = |Z|^{-1} \text{mech} c^2 \omega_n. \quad (III, 6) \]
Only if the magnetic field strength coincides with one of these discrete values \( B_n \), radiationless circular trajectories are possible. In all other cases, where \( B \neq B_n \), the particle spirals inwards as expected under emission of radiation until it comes to rest in

Fig. 2.—5. Particle trajectory in the resonant case \( B = B_1 \). Parameters are unchanged with respect to Figure 1. The initial velocity \( u \) is \( u = 0.2 \) (Fig. 2); \( u = 2.8 \) (Fig. 3); \( u = 4.5 \) (Fig. 4) and \( u = 20 \) (Figure 5).
a certain point dependent from the initial conditions. Figure 1 shows a numerical solution of this sort.

It does, however, not seem to be worthwhile studying the difference in energy loss if compared with the corresponding results of the Lorentz-Dirac theory or other theories of radiation reaction (e.g. Mo-Papas\textsuperscript{9}). All these equations reduce in lowest order to the form\textsuperscript{10}

\[ m c^2 \ddot{u}^n = K^n + \frac{3}{5} r_0^2 \left( - F^0 F_1, u_r + (F^0 F_1 u, u_0) u_0^0 \right) \]

which was recently proposed as an exact equation\textsuperscript{11}. The classical corrections for the energy loss following from this equation are extremely small and commonly masked by quantum effects\textsuperscript{9,12,13}.

On the other hand, the resonant field strengths are extremely high and hardly attainable in a terrestrial experiment. For a rough evaluation we can put\textsuperscript{14}

\[ B_1 \approx |Z|^{-1} r_0^{-1} 2 \pi m_{\text{exp}} c^2 \quad (\text{III, 7}) \]

and one recognizes this as the same magnitude as an electric field strength would have “on the surface” of a spherical electron with classical electron radius \( r_0 \). But it seems plausible that the external field strength must come in the order of magnitude of the field strengths governing the interior of the electron, so as to produce a “resonance” with the internal modes.

For a numerical demonstration of how the particle approaches the stationary final state, we have chosen \( n=1 \) ( \( \Rightarrow 1 \) circular revolution per proper time interval \( \Delta s \)) and we have shot in the particle with constant velocity \( u \) \( \equiv \left| \frac{\mathbf{v}}{c} \cdot (1 - \mathbf{v}^2/c^2)^{-1/2} \right| \) as initial condition in the first \( \Delta s \)-interval. Figures 2 to 5 show a plot of the trajectories for a wide range of initial velocities \( u \). The dashed circle would be followed by a non-radiating particle with same experimental mass \( m_{\text{exp}} \) and same initial condition, according to Eq. (II, 12) with \( m_{\text{mech}} \) replaced by \( m_{\text{exp}} \) \((= m_{\text{mech}} + m_{\text{el}})\).

From these figures, one recognizes that the radius \( A \) of the final circle depends strongly from the initial condition \( u \). The magnitude of the final radius is indeed a crucial test to our interpretation that the final radiationless motion be a sort of “internal resonance” of the extended particle. In order that this interpretation be true, the final radius \( A \) should not exceed the size \( (\Delta s) \) of the particle so that the motion can be viewed as an internal rotation. Figure 6 shows a plot of numerically calculated final radii as function of the initial velocity \( u \); and one recognizes indeed: \( A < \Delta s \), even for very high impact velocities \( (u = 10^3 \approx u_{\text{in}}^0 = (E/m_{\text{exp}} c^2) \).

Next, one would like to know the energy remaining in the circular motion. Obviously, the whole initial energy \( \mathcal{E}_{\text{in}} = m_{\text{exp}} c^2 u_{\text{in}}^0 \) cannot be radiated away because of that final energy connected with the circular motion. The latter part of the energy is [observe the first of (III, 1) and (III, 2)]

\[ \mathcal{E}_{\text{fin}} = m_{\text{mech}} c^2 u_{\text{fin}}^0 = m_{\text{mech}} c^2 b^* \]

\[ = \frac{m_{\text{mech}} c^2}{m_{\text{exp}}} \left( 1 + (2 \pi n A \Delta s)^{-2} \right) \quad (\text{III, 8}) \]

so that the fraction \( (e_{\text{rad}}) \) of the radiated energy becomes

\[ e_{\text{rad}} = \frac{\mathcal{E}_{\text{in}}^{-1}}{\mathcal{E}_{\text{fin}}^{-1}} (\mathcal{E}_{\text{in}} - \mathcal{E}_{\text{fin}})^{-1} (u_{\text{in}}^0 - u_{\text{fin}}^0) \]

\[ = \frac{m_{\text{mech}}}{m_{\text{exp}}} \frac{1}{1 + (2 \pi n A \Delta s)^{-2}} \quad \text{III, 9} \]

This quantity is plotted for \( n=1 \) in Figure 7, and one recognizes that for low impact velocities \( (u \to 0) \) the final energy equals roughly the constant fraction \( 1 - m_{\text{mech}}/m_{\text{exp}} \) whereas for high impact velocities the most part of the initial energy is radiated away \( (e_{\text{rad}} \to 1) \). So we see that, in the non-relativistic limit, that part of the initial energy \( \mathcal{E}_{\text{in}} \) \((= \mathcal{E}_{\text{mech, in}} + \mathcal{E}_{\text{el, in}})\) is radiated away, which has been carried along by the electromagnetic subsystem alone \( \mathcal{E}_{\text{el, in}} = m_{\text{el}} c^2 u_{\text{in}}^0 \)

\[ \mathcal{E}_{\text{rad}} = \mathcal{E}_{\text{el, in}} ; \quad \mathcal{E}_{\text{fin}} = \mathcal{E}_{\text{mech, in}} \quad \text{III, 10} \]

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**Fig. 6.** Plot of the (reduced) final radius \( A/\Delta s \) versus relativistic velocity \( u \). Solid curve: numerical calculation for the relativistic case. Dashed curve: Exact result (V, 20) for the non-relativistic case.
initial condition \( \{ \mathbf{v}_{\text{in}}(t) ; t \in [0, \tau_0] \} \) is representable as an exponential series
\[
v(t) = \sum_{r=0}^{\infty} c_r \exp(\hat{\lambda}_r t),
\]
where the coefficients \( c_r \) are determined as follows
\[
c_r = \frac{P(\hat{\lambda}_r)}{h'(\hat{\lambda}_r)},
\]
the \( \hat{\lambda}_r \) \((r = 0, 1, 2, 3, \ldots) \) being the (complex) characteristic roots of the "characteristic equation"
\[
h(\hat{\lambda}) \equiv m_{\text{mech}} \hat{\lambda} + \tau_0^{-1} m_{\text{el}} (1 - \exp[-\hat{\lambda} \tau_0]) = 0.
\]
The initial condition enters the function \( P(\hat{\lambda}) \) explicitly in the form
\[
P(\hat{\lambda}) = m_{\text{mech}} \mathbf{v}_{\text{in}}(\tau_0) \exp[-\hat{\lambda} \tau_0]
\]
\[
+ (m_{\text{mech}} \hat{\lambda} + \tau_0^{-1} m_{\text{el}}) \int_0^{\tau_0} \mathbf{v}_{\text{in}}(t) \exp[-\hat{\lambda} \tau] \, d\tau.
\]
From the series representation (IV, 3) it is immediately obvious that the existence of runaway solutions depends from the existence of roots \( \hat{\lambda}_r \) with positive real part. It was already shown in an earlier paper that for Eq. (IV, 2) all characteristic roots have non-positive real parts, and the only one with vanishing real part \([\text{Re}(\hat{\lambda}_0) = \alpha_0 = 0]\) has also vanishing imaginary part \([\text{Im}(\hat{\lambda}_0) = \beta_0 = 0]\). This zero root corresponds to a pure translational mode (observe that the free particle can move with arbitrary constant velocity \( \mathbf{v} \)), whereas all others modes \((\hat{\lambda}_r = a_r + ib_r ; r = 1, 2, 3, \ldots) \) are damped: \( a_r < 0 \).
Obviously, these damped modes characterize the internal vibrations of the extended particle. The damping of the modes arises from the irradiation of electromagnetic energy.

As already pointed out by Wildermuth, this point of view meets with a certain difficulty. Suppose that the length parameter \( \Delta s \) (or \( \tau_0 \)) tends to zero, which means that we make the size of the particle arbitrarily small. Holding the total charge \( (Z) \) fixed in this process, we see from (II, 15) that the electromagnetic mass \((m_{\text{el}})\) diverges to infinity, but since the experimental mass \((m_{\text{exp}})\) is finite and is the sum of the mechanical and electromagnetic part, the first one \((m_{\text{mech}})\) must become negative. This mechanism creates runaway solutions and the acausal behaviour, as was shown earlier by Wildermuth and more recently by Moniz and Sharp. It is then argued that the negative value of the mechanical mass...
is also the origin of the runaway solutions in the Lorentz-Dirac point approximation\textsuperscript{15}.

However, as long as we adopt $m_{\text{mech}} > 0$, all those internal modes are damped, and this is true for both the special case (IV, 2) and the general class (IV, 1), as can be readily shown: Trying the ansatz (IV, 3) for (IV, 1) yields after separation into real and imaginary parts ($\lambda_r = a_r + i b_r$)

$$m_{\text{mech}} + m_{e} r \int f(\tau) \exp[-a_r \tau] \cos(b_r \tau) \, d\tau = 0$$

(IV, 7a)

$$\int_0^\infty f(\tau) \exp[-a_r \tau] \sin(b_r \tau) \, d\tau = 0.$$  \hspace{1cm} (IV, 7b)

But the second equation can only be fulfilled for non-zero $b_r$, if the pre-factor $(\int f(\tau) \exp[-a_r \tau])$ of the sin-function is not monotonically decreasing with respect to the variable $\tau$. Since $f(\tau)$ is monotonically decreasing due to (II, 8) and (II, 9), the damping constants $(a_r)$ must be all negative in order that (IV, 7b) can be satisfied. Hence, all modes are damped ($a_r < 0, \forall r$).

The conclusion that there are no runaway solutions as long as $m_{\text{mech}} > 0$, holds also for the relativistic domain. Rewriting Eq. (II, 5) as

$$m_{\text{exp}} c^2 u_0^0 + m_{e} c^2 [Q^0 - u_0^0] - c^2 \int_0^\tau R \, dt + m_{\text{exp}} u_0^0 + m_{e} c^2 [Q^0 - u_0^0] = 0,$$

we see that the change of kinetic energy ($m_{\text{exp}} c^2 u_0^0$) and of internal distortion energy ($m_{e} c^2 [Q^0 - u_0^0]$) is balanced by the emitted power ($c^2 \int_0^\tau R \, dt$). But since the radiation rate $R$ is positive-semidefinit, the right-hand side of (IV, 8) is monotonically decreasing during the motion; however, as long as $m_{\text{mech}} > 0$, the left-hand side of (IV, 8) is always positive on account of $u_0^0 \geq 1; Q^0 \geq 0 (f(\tau) \geq 0)$. Thus, the radiated power cannot become infinite for $t \to \infty$ but must assume a finite value

$$c^2 \int_0^\infty R \, dt = C_{\text{rad}} < \infty.$$  \hspace{1cm} (IV, 9)

This is only possible, if

$$\lim_{t \to \infty} \frac{\dot{R}}{\tau_0} = - m_{e} c [\dot{Q} u_0] = 0.$$  \hspace{1cm} (IV, 10)

and from the considerations below (II, 9) then follows

$$\lim_{t \to \infty} u^{0}_{(\tau)} = \text{const}.$$  \hspace{1cm} (IV, 11)

which completes the proof of non-existence of runaway solutions in the case $m_{\text{mech}} > 0$. Obviously, this proof is valid whenever the external force does not supply energy to the particle, especially if no external force is present. Thus, the earlier proof\textsuperscript{18} for the special case $f(\tau) = 1$ and no force present is generalized now to arbitrary $f(\tau)$ from (II, 9). (Observe here:

$$\Re \equiv 0 \Rightarrow \dot{Q}_0 \equiv 0 \Rightarrow m_{\text{mech}} c^2 \dot{u}_0^0 = K^2 \equiv 0 \Rightarrow u_0^0 = \text{const}.$$

V. Non-relativistic Motion in a Constant Magnetic Field

Now we proceed to study the non-relativistic transverse motion in a constant, homogeneous $B$-field. Especially, we are interested here in an exact determination of the radiationless final state of motion for a given initial condition.

Starting with the general equations of motion (IV, 1)

$$m_{\text{mech}} \frac{dv^1(t)}{dt} + m_{e} \int_{\tau=0}^{\tau_0} f(\tau) \frac{dv^1(t-\tau)}{d\tau} \, d\tau = 0,$$

(IV, 1a)

and unifying the two Eqs. (V, 1) into a single one

$$m_{\text{mech}} \frac{dv^1(t)}{dt} + m_{e} \int_{\tau=0}^{\tau_0} f(\tau) \frac{dv^1(t-\tau)}{d\tau} \, d\tau = 0,$$

(IV, 1b)

it is convenient to introduce the complex velocity ($Q = m_{e} c/m_{\text{mech}}$)

$$v_{(\tau)} = v^1(t) + i v^2(t)$$

(V, 2)

and the cyclotron frequency $\Omega$

$$\Omega = (m_{\text{mech}} c)^{-1} |Z| B > 0$$

(V, 3)

where $Z = 1$. Thus, the complex velocity $v_{(\tau)}$ can be written as

$$v_{(\tau)} = Q [\Omega v_{(\tau)}].$$

(V, 4)

and from the considerations below (II, 9) then follows

$$\lim_{t \to \infty} u^{0}_{(\tau)} = \text{const}.$$  \hspace{1cm} (IV, 11)

which completes the proof of non-existence of runaway solutions in the case $m_{\text{mech}} > 0$. Obviously, this proof is valid whenever the external force does not supply energy to the particle, especially if no external force is present. Thus, the earlier proof\textsuperscript{18} for the special case $f(\tau) = 1$ and no force present is generalized now to arbitrary $f(\tau)$ from (II, 9). (Observe here:

$$\Re \equiv 0 \Rightarrow \dot{Q}_0 \equiv 0 \Rightarrow m_{\text{mech}} c^2 \dot{u}_0^0 = K^2 \equiv 0 \Rightarrow u_0^0 = \text{const}.$$
\[ 1 + \frac{\partial}{\partial r_0} \int_0^{r_0} dr f(t) \exp[-a r] \cos(b r) = \left(a^2 + b^2\right)^{-1} b \Omega, \quad (V, 7a) \]

\[ a \int_0^{\tau_0} dr f(t) \exp[-a r] \sin(b r) = \left(a^2 + b^2\right)^{-1} a \Omega. \quad (V, 7b) \]

From the last equation, one realizes easily that solutions for \( \lambda \) with vanishing real part \((a = 0)\): undamped oscillations) are only possible for the class of piecewise constant structure functions \( f(t) \), such as described under (II,13) up to (II,18). This non-relativistic result corresponds to the relativistic one found in Section II.

Next, we specialize formulae (V, 1) (V, 3), and (V, 7) to the simple case \( f(t) \equiv 1 \) obtaining as equation of motion

\[ \frac{db(t)}{dt} = \tau_0^{-1} [\psi(t) - \psi(t-\tau_0)] - i \Omega \psi(t) = 0 \quad (V, 8) \]

with the characteristic equation

\[ h(\lambda) \equiv \lambda + \tau_0^{-1} \varrho (1 - \exp[-\lambda \tau_0]) - i \Omega = 0 \quad (V, 9) \]

or split up into real and imaginary part \((\lambda = a + i b)\)

\[ \varrho + \tau_0 = \varrho \exp[-a \tau_0] \cos(b \tau_0) \quad (V, 10a) \]

\[ -(b \tau_0 - \Omega \tau_0) = \varrho \exp[-a \tau_0] \sin(b \tau_0). \quad (V, 10b) \]

From these equations it is seen easily that only \( a \leq 0 \) is possible. Moreover, \((a, b) = (0, 0)\) is no longer a solution as was the case for vanishing external force \( (\Omega = 0) \).

In order to study the stationary modes characterized by a root of the form

\[ (a, b) = (0, b_n) ; \quad n = 1, 2, 3, \ldots, \]

one eliminates the frequency \( b \) from the set of Eqs. (V,10) and finds

\[ \Omega \tau_0 = n \cdot 2 \pi \pm \arccos \left[ \frac{a \tau_0 + \varrho}{\varrho \exp[-a \tau_0]} \right] \quad (V, 11) \]

\[ \pm \sqrt{1 - \left( \frac{a \tau_0 + \varrho}{\varrho \exp[-a \tau_0]} \right)^2} \varrho \exp[-a \tau_0] \]

\[ n = 0, \pm 1, \pm 2, \pm 3, \ldots. \]

The two branches of the single curve (V,11) have been plotted in a \((\tau_0/\Omega)\)-diagram (Fig. 8), and one recognizes very well from this figure, that the infinite set of damping constants \( \alpha_r (r = 0, 1, 2, 3, \ldots) \) for arbitrary but fixed \( \Omega \) can be obtained by simply shifting a single curve (V,11) (e.g. that for \( n = 0 \)) parallel to the \((\Omega \tau_0)\)-axis by the amount \( n \cdot 2 \pi \). Therefore, increasing the value of \( \Omega \) from zero, the real parts of the modes \( (\lambda_r) \) become zero successively so that for \( \Omega_n \tau_0 = n 2 \pi \) there exists always one mode with zero real part \((\alpha_n = 0)\).

Thus we arrive at an infinite set of radiationless motions with the corresponding cyclotron frequencies given by \( \Omega_n \). Observe that the intersections of two curves (V,11) for different \( n \) do not necessarily indicate a degeneracy, because the corresponding imaginary parts \( b_r \) are in general different there.

Our next task is now to find the final state of motion for a given initial condition. To this end, let

Fig. 8. The first three damping constants \((\alpha_1, \alpha_2, \alpha_3)\) as function of the frequency \( \Omega \). The branches with positive ascent at \( \Omega = 0 \) belong to modes with negative frequency \( b \). These modes can never come to resonance because of their opposite sense of rotation.
us first point out that the equation of motion \( (V, 8) \) has a quite similar structure to the force-free case \((IV, 2)\); only the constant pre-factors of the (complex) velocity are different. We can therefore again apply the Laplace transform method resulting in the series representation \((IV, 3)\), where the formulae for determining the coefficients \(c_i\) must be altered correspondingly to the new Equation \((V, 8)\):

\[
P_{c(i)} = v_{\text{in}}(t_0) \exp[-\lambda t_0]
+ (\lambda + \tau_0^{-1} Q - i \Omega) \int_0^{\tau_0} v_{\text{in}}(t) \exp[-\lambda t] \, dt,
\]

whereas the derivative \(h'(z)\) is the same as before \(\text{[apart from dividing it by } m_{\text{mech}}\text{, see } (V, 9)\text{]}\). Since all terms in the series \((IV, 3)\), which contain characteristic roots \((\lambda_i)\) with negative real part \((a_i)\), are dying out for \(t \to \infty\), the only surviving term is \(c_0\), which belongs to a root with vanishing real part \((a_0 = 0)\). We know already that such a term occurs if \(\Omega\) assumes one of the above values \(\{\Omega_n\}\), and then the frequency \(b_0\) of the final stationary motion becomes identical to \(\Omega_n\). Hence, the asymptotic form of the solutions is

\[
\lim_{t \to \infty} v(t) = \left(1 + \varrho\right)^{-1} [v_{\text{in}}(t_0) + \tau_0^{-1} \varrho]
\cdot \int_0^{\tau_0} v_{\text{in}}(t) \exp[-i \Omega_n \tau] \, d\tau \cdot \exp[i \Omega_n \tau].
\]

From this formula one recognizes quite clearly that the final state of motion is determined from the velocity \(v_{\text{in}}(t_0)\) at the end of the initial period and from the Fourier component of \(v_{\text{in}}(t)\) at the cyclotron frequency \(\Omega_n\). So we see that the external force oscillating with frequency \(\Omega_n\) selects the internal mode of same frequency, whereas all other "non-resonant" modes are dying out (Figure 9).

However, we know already from the previous sections that for a general structure function \(f(\varnothing)\) from (II, 9) such a resonance phenomenon is not possible. We conclude that such a particle exhibits a certain rigidity (in the relativistic sense) and resembles therefore a sort of "point particle", if one does not associate with this notion the vanishing size but rather thinks of the impossibility to excite interior vibrations (cf. the quite similar findings of Moniz and Sharp for their "point particle limit" \(^1\)).

Let us now specialize the initial condition as \(v_{\text{in}}(t_0) = v_0 = \text{const}\), which we had already used for our numerical calculations in Section III. For this initial condition, one concludes from \((V, 13)\)

\[
\lim_{t \to \infty} v(t) = v_0 \left(1 + \varrho\right)^{-1} \exp[i \Omega_n t]
= m_{\text{mech}} v_0 \exp[i \Omega_n t].
\]

Defining the absolute value of the non-relativistic total momentum by

\[
P_{(t)} := m_{\text{exp}} |v(t)|
\]

and its final value by \(P_\infty\), one verifies from \((V, 14)\)

\[
P_\infty = m_{\text{mech}} |v_0| = P_{\text{mech}, \text{in}},
\]

so that the irradiated part becomes \(P_{\text{rad}} := m_{\text{el}} |v_0|\) and therefore

\[
P_\infty + P_{\text{rad}} = P_{\text{in}} = m_{\text{exp}} |v_0| = P_{\text{mech}, \text{in}} + P_{\text{el}, \text{in}}.
\]

This confirms the numerical result of Sect. III, that only that part of the energy dragged along initially by the electromagnetic mass \((m_{\text{el}})\) alone, is radiated away; whereas the mechanical part is stored in the final motion. This statement can be generalized to an

Fig. 9. The lowest positive frequencies \(b_i(r=0, 1, 2, 3, 4)\) as function of \(\Omega\). Whenever \(\Omega\) equals the frequency of an internal mode \((\text{dashed line: } b_\varnothing = \Omega_\varnothing)\), this mode becomes undamped. The frequencies \(b_{r}\) are almost constant with exception of the resonance region, where a characteristic transition from one constant value to the other one is performed.

\[
\lim_{t \to \infty} v(t) = v_0 \left(1 + \varrho\right)^{-1} \exp[i \Omega_n t]
= m_{\text{mech}} v_0 \exp[i \Omega_n t].
\]
arbitrary initial condition and one finds from (V, 13)
\[ \Psi_{\infty}(t) = m_{\text{exp}} v_{\infty}(t) = \left[ \frac{\Psi_{\text{mech}, \text{in}(t_0)} + \int_{t_0}^{t} \Psi_{\text{mech}, \text{in}(\tau)} d\tau}{\tau_0} \right] \exp \left[ -i \Omega_n t \right] \cdot \exp \left[ i \Omega_n t \right]. \quad (V, 18) \]

This means that in addition to \( \Psi_{\text{mech}, \text{in}(t_0)} \) the Fourier component of \( \Psi_{\text{mech}, \text{in}(T)} \) at the resonance frequency \( \Omega_n \) is preserved, whereas the energy contained in \( \Psi_{\text{els}(t_0)} \) and in the non-resonant modes is radiated away.

The final problem to be solved in this section is the determination of the stationary radius (\( A \)). Contrary to the relativistic case, where the radius of the final circular motion is available only numerically, we can integrate here Eq. (V, 14) and find
\[ \lim_{t \to \infty} \delta(t) := \lim_{t \to \infty} \int v(\tau) d\tau = v_0 \left| (\Omega_n [1 + q])^{-1} \right| \exp \left[ i \Omega_n t \right]. \quad (V, 19) \]

Hence, the stationary radius (\( A \)) is
\[ A_n = \left| v_0 \right| (\Omega_n [1 + q])^{-1} = \left( 2 \pi n [1 + q] \right)^{-1} \tau_0 |v_0|. \quad (V, 20) \]

The linear dependence of \( A \) from the initial velocity \( |v_0| = u (1 + u^2)^{-1/2} \) has also been entered in Figure 6. It becomes clear from this figure that the non-relativistic result (V, 20) is not too bad in comparison to the relativistic calculations (apart from a certain resonance character of the latter one), so that one can use (V, 20) for the following evaluations:

The maximal energy stored in the final circular motion is bounded for \( u \to \infty \) (\( \to |v_0| \to c \)) according to (III, 8) and (V, 20)
\[ E_{\text{fin}} = m_{\text{mech}} c^2 \sqrt{1 + (2 \pi n A_n \Delta s^{-1})^2}, \]

\[ \approx m_{\text{mech}} c^2 \sqrt{1 + (1 + q)^{-2}} \left| v_0 \right|^{2/3} \]
\[ E_{\text{fin}, \text{max}} \approx m_{\text{mech}} c^2 \sqrt{1 + (1 + q)^{-2}} \leq \sqrt{2} m_{\text{exp}} c^2. \quad (V, 21) \]

Hence, there can be stored up to 40% of the rest energy of the particle in the finale stationary state.

VI. The Longitudinal Motion

So far, we have restricted our considerations to the pure transverse motion, which is limited to a plane orthogonal to the magnetic lines of force. In this context, one is interested in the question, how an initial additional component of the velocity parallel to the magnetic field develops in time for both cases where \( \omega \) is equal or unequal to one of the resonance frequencies \( n 2\pi \Delta s^{-1} \).

The analogous problem for the Lorentz-Dirac and for the Mo-Papas equation has already been solved and it is found in both theories that an initial longitudinal component of 3-velocity is a constant of the motion. The energy loss due to radiation is taken here completely from the transverse velocity. Under certain initial conditions, the same result holds also for the general class (I, 1), as can be seen easily from the zero and third component of the equation of motion:
\[ m_{\text{mech}} c^2 \dot{v}^3 + m_{\text{el}} c^2 \{ \dot{Q} - (\dot{Q} \dot{u}) u \} = 0, \quad (VI, 1a) \]
\[ m_{\text{mech}} c^2 \dot{u}^3 + m_{\text{el}} c^2 \{ \dot{Q}^3 - (\dot{Q} \dot{u}) u^3 \} = 0. \quad (VI, 1b) \]

Putting here
\[ v^3(s) = c^{-1} v^3 u^3(s); \quad v^3 = \text{const} \quad (VI, 2) \]
reduces the second Eq. (VI, 1b) to the first one (VI, 1a). Hence, if the longitudinal component \( v^3 \) is constant in the first \( \Delta s \)-interval, then it will remain constant throughout the motion.

However, if there is a non-vanishing acceleration in the first \( \Delta s \)-interval, then \( v^3(s) \) will not be constant. For numerical calculations in the relativistic case (I, 1), we have chosen \( dv^3(s)/ds = \dot{v}^3 \) in, = const and the initial condition for the transverse component \( v^3(s) \) was chosen to be constant: \( v^3(s) = \text{const} \). We have always observed the same qualitative behaviour of the longitudinal component \( v^3(s) \), no matter whether there existed resonance or not. Moreover, no serious difference was observed between the relativistic case and the non-relativistic limit. Figure 10 shows a typical result: within a proper

![Fig. 10. Ordinary velocity \( v^3 \) versus laboratory time \( T := c t/J s_{\exp} \). After a characteristic time period of some \( \tau_0 \)-intervals, the longitudinal velocity \( v^3 \) assumes a constant value.](image-url)
time interval $\Delta t$ the longitudinal component $v_3^{(s)}$ is damped down to a constant value $v_3^\infty$.

In the non-relativistic limit, we can again calculate exactly the final value $v_3^\infty$, because the equations (V, 1) (for $f_0(\tau) \equiv 1$) are decoupled and the 3-component reads

$$m_{\text{mech}} \frac{dv_3(t)}{dt} + m_{\text{el}} \tau_0^{-1} (v_3(t) - v_3(t-\tau_0)) = 0. \quad (V I, 3)$$

Hence, we can apply formulae (IV, 3) to (IV, 6) with the characteristic root $\lambda_0$ being equal to zero (pure translation in the final state):

$$v_3^\infty \equiv c_0 = \lim_{t \to \infty} (1 - \frac{1}{2} m_0 / m_{\text{el}} c^2) \cdot \lambda_0. \quad (VI, 4)$$

So we see that the longitudinal velocity is also a constant of the motion, if one regards time intervals much larger than $\tau_0$ (see Figure 10).

Acknowledgement

The authors are indebted to Prof.s J. Petzold and W. Weidlich for stimulating discussions.

8. T. Erber, Fortschr. Phys. 9, 343 [1961].