Scattering of a Polarized Photon by a Polarized Electron

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The scattering of a polarized photon by a polarized free electron is completely calculated in covariant form using Clifford’s alternating vector algebra. The result is compared with several special results of the existing literature.

Introduction

The scattering of a polarized photon by a polarized electron has completely been calculated only for the primary electron at rest in a three-dimensional vector form\(^1\).\(^2\); especially the spin of the secondary electron has not been expressed in a covariant way. In consequence of that, the formulae get a compact form only by the introduction of suitably chosen mirror spins. If the result is explicitly expressed by the original spins and by Stokes parameters, the formulae become clumsy and unelegant\(^2\). Therefore it seems useful to recalculate the scattering in a covariant form. As will be shown, this can be done in a rather convenient way by using consistently Clifford’s alternating algebra and by a suitable choice of the electromagnetic gauge.

§ 1. Scattering Formula

The essential part of the differential cross-section for Compton effect is given by

$$X_0 = \left| \bar{u}(p_2, a_2) F_c u(p_1, a_1) \right|^2$$ \hspace{1cm} (1.1)

where

$$F_c = a_{2}\mu p_1 + k_1 a_1 - \mu p_1 - k_2 a_2^{*}.$$ \hspace{1cm} (1.2)

For reference see f. i. Akhieser-Berestetskii\(^3\), § 25.\(^\dagger\):

\(p_1, p_2\) are the four-momenta of the electrons, \(k_1, k_2\) those of the photons; \(a_1, a_2, a_3, a_4\) are the four-vectors of polarization of the four particles\(^4\). We take the square of a time-like vector positive, of a space-like one negative. For the propagation vectors we have

$$p_1 \cdot p_1 - p_2 \cdot p_2 = \mu^2; \hspace{0.5cm} (\mu \equiv m c); \hspace{0.5cm} k_1 \cdot k_1 - k_2 \cdot k_2 = 0.$$ \hspace{1cm} (1.3)

The four-vectors of polarization are orthogonal to the respective propagation vectors:

$$p_1 \cdot a_1 = p_2 \cdot a_2 = k_1 \cdot a_4 = k_2 \cdot a_2 = 0.$$ \hspace{1cm} (1.4)

Therefore they all are space-like, and as unit vectors they have the squares

$$a_1 \cdot a_1 = a_2 \cdot a_2 = a_3 \cdot a_3 = a_4 \cdot a_4 = -1.$$ \hspace{1cm} (1.5)

Conservation of four-momentum implies a twofold expression for the momentum transfer \(K\) from the electron-to the photon-system:

$$K = p_1 - p_2 = k_2 - k_1.$$ \hspace{1cm} (1.6)

For the products of vectors we use the convention: dot for scalar product, no dot for Clifford multiplication (see § 2). For the Dirac amplitudes we have the eigenvalue equations

$$(p - \mu) u(p, o) = 0; \hspace{0.5cm} \bar{u}(p, o) (p - \mu) = 0.$$ \hspace{1cm} (1.7)

$$\sigma\gamma_5 - 1 \ u(p, o) = 0; \hspace{0.5cm} \bar{u}(p, o) (\sigma\gamma_5 - 1) = 0.$$ \hspace{1cm} (1.8)

\(\sigma\gamma_5\) is the pseudovector of spin, \(\sigma p\gamma_5 / \mu\) the antisymmetric spin tensor. As a consequence of the eigenvalue equations and of \(\bar{u}\gamma_5 = u^{*}\) the bi-spinor \(\pm u\bar{u}\) is the projector:

$$\pm u(p, o) \bar{u}(p, o) = \frac{\mu + p}{2 \mu} + \frac{1 + \sigma\gamma_5}{2}.$$ \hspace{1cm} (1.9)

§ 2. Clifford Algebra

The Clifford vector algebra, of which the Dirac algebra is the four-dimensional special case, is an associative algebra of alternating products of vectors, for which we use the symbol

$$\{v_1 v_2 \ldots v_n\}.$$ \hspace{1cm} (2.1)

The associative product of two alternating products is given by

$$\{v_1 \ldots v_m\} \{w_1 \ldots w_n\} = \{v_1 \ldots v_m w_1 \ldots w_n\} + \text{contractions}.$$ \hspace{1cm} (2.2)
The contractions are meant in the sense of the ordering theorem\textsuperscript{6} for alternating quantities, the contraction of two vectors being the scalar product:

\[ \text{contr} \{ v w \} = v \cdot w. \]

(2.3)

The simplest special case of Eq. (2.2) is the Clifford product of two vectors:

\[ v w = \{ v w \} + v \cdot w. \]

(2.4)

For the unit vectors \( \gamma_0, \gamma_1, \gamma_2, \gamma_3 \) this yields the wellknown Dirac rules: \( \gamma_j \gamma_k \) alternating for \( j + k \); \( \pm 1 \) for \( j = k \) timelike resp. spacelike. — Two further special cases of Eq. (2.2) are

\[ \{ v iv \} x = \{ v w x \} + (v w - w v)'x; \]

(2.5)

\[ \{ v w \} \{ x y \} = \{ v w x y \} + (v w - w v)'(xy - y x) + (v w - w v)'xy. \]

(2.6)

In fourdimensional space we have alternating products only up to four factors. Any four-factor alternating product is proportional to the four-dimensional unit volume characterized by the quantity

\[ \gamma_5 = \gamma_0 \gamma_1 \gamma_2 \gamma_3 = i(\gamma_0 \gamma_1 \gamma_2 \gamma_3); \quad \gamma_5 \gamma_5 = 1. \]

(2.7)

In consequence the following expression is a pure number

\[ \{ v w x y \} \gamma_5 = \frac{1}{i} [v w v w]. \]

(2.8)

The last symbol means the determinant

\[ [v w x y] = \begin{vmatrix} v_0 & v_1 & v_2 & v_3 \\ w_0 & w_1 & w_2 & w_3 \\ x_0 & x_1 & x_2 & x_3 \\ y_0 & y_1 & y_2 & y_3 \end{vmatrix}. \]

(2.9)

For practical calculations the following formula for \( n \) linearly dependent vectors \( v_1 \ldots v_n \) is important.

\[ v_n \{ v_1 \ldots v_{n-1} \} - v_1 \{ v_2 \ldots v_n \} + v_2 \{ v_3 \ldots v_n \} + \ldots + v_{n-1} \{ v_n \ldots v_1 \}. \]

(2.10)

The validity of this formula is based on the fact that the difference of the left and right hand sides is alternating in all the \( n \) dependent vectors. The symbol \( \{ \} \) hereby can indicate any alternating expression in the \( n-1 \) vectors.

\section*{§ 3. Description of Propagation and Polarizations}

We shall describe the dynamics of the scattering process by three vectors \( K, k, p \); the momentum transfer \( K \) is already defined by Eq. (1.6), \( k \) and \( p \) are the internal momenta of the photon- resp. electron-systems:

\[ k \equiv k_2 + k_1; \quad p \equiv p_1 + p_2. \]

(3.1)

The squares of these three vectors are related by

\[ k^2 = -K^2 = p^2 - 4m^2. \]

(3.2)

The two internal momenta are perpendicular to \( K \)

\[ k \cdot K = 0; \quad p \cdot K = 0. \]

(3.3)

The inversion of Eqs. (1.6) and (3.1) is

\[ k_1 = \frac{k - K}{2}; \quad k_2 = \frac{k + K}{2}; \quad p_1 = \frac{p + K}{2}; \quad p_2 = \frac{p - K}{2}. \]

(3.4)

For the spin unit vectors we have the relations

\[ p \cdot \sigma_1 = -K \cdot \sigma_1; \quad p \cdot \sigma_2 = K \cdot \sigma_2. \]

(3.5)

In order to introduce Stokes parameters to describe the polarization of a photon (see Akhieser, Berestetskii\textsuperscript{3}, § 2) we have to choose two perpendicular polarization unit vectors \( \alpha, \beta \) to arrive at the following dyadic form of the light polarization density matrix:

\[-a = \frac{1}{2} \begin{pmatrix} 1 + \xi & \alpha \beta + \eta \delta \beta \\ \alpha \delta - \beta \delta & \alpha \beta - \eta \delta \beta \end{pmatrix}. \]

(3.6)

To make the description simple and independent of frame, there is only one choice for the unit vectors \( \alpha, \beta \): We take both of them perpendicular to both the propagation vectors \( k_1, k_2 \), and one of them — say \( \alpha \) — in addition perpendicular to \( p \):

\[ \alpha^2 = -1; \quad \beta^2 = -1; \quad \alpha \cdot k_j = 0; \quad \beta \cdot k_j = 0; \quad \alpha \cdot p = 0. \]

(3.7)

As a consequence of momentum conservation \( \alpha \) is also perpendicular to \( p_1 \) and \( p_2 \).

\( \alpha \) and \( \beta \) are polarization vectors of both the primary and the secondary photon. They are space vectors in the “brick-wall system” of Breit; in this frame the two photons have equal frequencies and propagate into opposite directions. — By (3.7) the unit vectors \( \alpha \) and \( \beta \) are determined apart from sign. We fix the signs too by demanding explicitly

\[ \beta = \frac{p - k \cdot k}{\sqrt{(p \cdot k)^2 - k^2}}; \quad [K k \alpha \beta] = K^2. \]

(3.8)
The two-dimensional polarization plane \( \alpha, \beta \) is perpendicular to the light propagation plane \( K, k \). The unit matrix of four-dimensional space can be expressed in dyadic form by the four perpendicular unit vectors

\[
I = \frac{kk + KK}{k^2} - \alpha \beta.
\]

From (3.8) we get for an arbitrary vector \( v \):

\[
v \cdot \alpha = \frac{1}{k^2} [Kkv\beta].
\]

For two vectors \( v, w \) we get

\[
vw \cdot (\alpha \beta - \beta \alpha) = -\frac{1}{k^2} [Kkvw].
\]

This equation follows directly from (3.8); to get it from (3.10), one has to apply Eq. (2.10) to three dependent vectors of the polarization plane.

The inversion of Eq. (3.6) reads

\[
\zeta = (\alpha \beta - \beta \alpha) \cdot \alpha \alpha^*; \quad \eta = (\alpha \beta + \beta \alpha) \cdot \alpha \alpha^*; \\
\zeta = i (\alpha \beta - \beta \alpha) \cdot \alpha \alpha^*.
\]

These expressions are independent of the gauge of \( a \), as the principal unit vectors \( \alpha, \beta \) are perpendicular to both propagation vectors \( k_1, k_2 \). By virtue of Eqs. (3.8) – (3.11) we have explicitly

\[
\xi = -I + \frac{kk - KK}{k^2} + 2 \left( \frac{p - p \cdot k}{k^2} k \right) \cdot \alpha \alpha^*;
\]

\[
\eta = \frac{1}{p^2 k^2 - (p \cdot k)^2} \left( Kpk \left( \frac{p - p \cdot k}{k^2} k \right) \cdot (\alpha \alpha^* + \alpha^* \alpha) \right); \\
\zeta = \frac{i}{k^2} [Kkaa^*].
\]

The helicity parameter \( \zeta \) is for photon 1 positive for positive helicity; for photon 2 the sign is opposite.

The expression \( F \) of Eq. (1.2) is gauge-independent (it does not change if a multiple of \( k_j \) is added to \( a_j \)). This can be used to simplify the algebra by taking the polarization vectors perpendicular to one of the electron propagation vectors, e.g. \( p_1 \):

\[
a_1 \cdot p_1 = a_2 \cdot p_1 = 0.
\]

This does not modify the Stokes parameters as defined by (3.12), but it alters the density matrix of each of the photon polarizations into

\[
-\bar{\xi}_j = a_j a_j^* = \frac{1 + \bar{\xi}_j}{2} \alpha \alpha + \frac{1 - \bar{\xi}_j}{2} \beta_j \beta_j + \frac{\eta_j + i \zeta_j}{2} \alpha \beta_j;
\]

\[
+ \frac{\eta_j - i \zeta_j}{2} \beta_j \alpha.
\]

As \( \alpha \) is already perpendicular to \( p_1 \), only the vector \( \beta \) has to be changed by a gauge transformation:

\[
\beta_j = \left( I - \frac{k_j p_1}{k_1 p_1} \right) \cdot \beta;
\]

\[\text{§ 4. Preparing the Scattering Formula}\]

The only easy way to evaluate the expression \( F \) of Eq. (1.2) seems following step by step the original 1938 paper. Firstly we use the gauge (3.14); as this violates reciprocity, we keep in mind that the final formula has to be invariant with respect to the following reciprocity transformation

\[
p \rightarrow p; \quad k \rightarrow k; \\
K \rightarrow -K; \quad a_1 \rightarrow a_2; \quad a_1 \rightarrow a_2.
\]

By virtue of the gauge chosen, \( a_j \) and \( p_j \) alternate. Therefore because of the right hand factor \( u_1 \) in (1.1) the terms \( \mu + p_1 \) cancel out, and we are left with

\[
-4p_1 \cdot k_1 p_1 \cdot k_2 F = a_2^* a_1 (k_2 k_1 + k_1 k_2) \\
+ \{a_2^* a_1\} (k^2 k + p \cdot k K).
\]

By means of Eq. (3.4) we get

\[
-4p_1 \cdot k_1 p_1 \cdot k_2 F = a_2^* a_1 (p \cdot k k + k_2 K) \\
+ \{a_2^* a_1\} (k_1 k + p \cdot k K).
\]

We now observe that by virtue of Eq. (1.7)

\[
\bar{u}_2 K u_1 = \bar{u}_2 (p_1 - p_2) u_1 = \bar{u}_2 (\mu - \mu) u_1 = 0.
\]

Therefore an arbitrary multiple of \( K \) can be added to (4.3), so we can substitute

\[
-4p_1 \cdot k_1 p_1 \cdot k_2 F = \left( \frac{p \cdot k}{k_1^2} - a_2^* a_1 + \{a_1 a_2^*\} \right) Q;
\]

\[\text{where}\]

\[Q \equiv k_1^2 k_2 \cdot p \cdot k K = 4 p_1 \cdot (k_1 k_2 - k_2 k_1).
\]
The vector \( Q \) is perpendicular to \( p_1 \), therefore we can transform \( Q_u Q = \frac{Q^2}{4} \left( 1 - \frac{p_1}{\mu} \right) (1 + s_1 \gamma_5) \); \( (4.6) \)

where \( s_1 \) is a "mirror-spin":

\[ s_1 = \sigma_1 - 2 \sigma_1 \cdot \frac{Q}{Q^2}; \quad s_1 \cdot s_1 = -1; \quad s_1 \cdot p_1 = 0. \quad (4.7) \]

This leads to the following expression for \( X \):

\[ X = \frac{4}{k^2 \left[ k^2 - (p \cdot k)^2 \right]} \bar{u}_2 \left( 1 - \frac{p_1}{\mu} \right) G(1 + s_1 \gamma_5) \bar{G} u_2, \quad (4.8) \]

where

\[ G = p \cdot k \sigma_2 \sigma_1 + k^2 \{ a_1 a_2^* \}, \]

\[ \bar{G} = p \cdot k a_1^* + k^2 \{ a_2 a_1^* \}. \quad (4.9) \]

\( p_1 \) commutes with all three quantities \( G, \bar{G}, s_1 \gamma_5 \), therefore in expression (4.8) the factor \( 1 - p_1/\mu \) can be shifted freely.

### § 5. Evaluation of the Scattering Formula

By straightforward application of Clifford multiplication (2.2) we get:

\[ G(1 + s_1 \gamma_5) \bar{G} = \left[ (p \cdot k)^2 a_1 a_2^* a_2 a_1^* + k^4 (1 - a_1^* a_2^* a_2 a_1) \right] (1 + s_1 \gamma_5) \]

\[ + p \cdot k k^2 \left[ \begin{array}{l}
(1 + s_1 \gamma_5) (a_2 a_2^* a_1 a_1^* + a_1 a_1^* a_2 a_2^*) \\
+ p \cdot k k^2 (s_1 (a_2 a_1^* - a_1 a_2) a_1 a_2 a_2 a_1^* + s_1 (a_1 a_1^* - a_1 a_2) a_2 a_1 a_1^*)
\end{array} \right] \]

\[ + k^4 (a_1 a_2^* - a_2 a_1^*) (a_2 a_2^* - a_2 a_1^*) (a_1 a_2 a_2 a_1^* - a_2 a_1 a_1 a_2^*) \cdot (a_1 a_2 a_2 a_1^* - a_2 a_1 a_1 a_2^*) \cdot s_1 \gamma_5 + s_1 \gamma_5 \]

\[ + k^4 (a_1 a_2^* - a_2 a_1^*) \cdot (a_2 a_2^* - a_2 a_1^*) (a_1 a_2 a_2 a_1^* - a_2 a_1 a_1 a_2^*) \cdot s_1 \gamma_5 \]

\[ (5.1) \]

If we insert that into Eq. (4.8), the factor \( 1 - p_1/\mu \) just enters all the alternating products of (5.1), as all vectors of Eq. (5.1) are perpendicular to \( p_1 \). So the only thing to do in order to get \( X \) is the evaluation of expressions \( \bar{u}_2 \{ \ldots \} u_2 \), where \( \{ \ldots \} \) indicates any alternating product of vectors. These expressions are traces of the product of \( \{ \ldots \} \) and \( u_2 \bar{u}_2 \) as given by Equation (1.9). Now all these traces are evaluated easily by observing that \( u_2 \bar{u}_2 \) has trace 1, and right- and left-hand eigenvalues 1 for vector \( p_2/\mu \), pseudovector \( \sigma_2 \gamma_5 \), tensor \( p_2 \sigma_2 \gamma_5/\mu \). It follows that \( \bar{u}_2 \{ \ldots \} u_2 \) may be \( \pm 0 \) only if \( \{ \ldots \} \) commutes with all three quantities. This yields the following traces for scalar and pseudoscalar quantities:

\[ \bar{u}_2 u_2 = 1; \quad \bar{u}_2 \gamma_5 u_2 = 0. \quad (5.2) \]

For vectors we have

\[ \bar{u}_2 a u_2 = (p_2/\mu) \cdot a; \quad \bar{u}_2 (a b c) \gamma_5 u_2 = (1/\mu) [p_2 a b c]. \quad (5.3) \]

For pseudovectors the result is

\[ \bar{u}_2 a \gamma_5 u_2 = -\sigma_2 \cdot a; \quad \bar{u}_2 (a b c) u_2 = (1/i) [a b c \sigma_2]. \quad (5.4) \]

Finally we get for tensors

\[ \bar{u}_2 \{ a b \} u_2 = \frac{1}{\mu} \left[ a b p_2 \sigma_2 \right]; \quad \bar{u}_2 \{ a b \} \gamma_5 u_2 = a b \cdot p_2 \sigma_2 - a_2 p_2 \mu. \quad (5.5) \]

From Eqs. (4.8) and (5.1) we get in this way

\[ \frac{\mu^2}{2} \left[ (p \cdot k)^2 - k^4 \right] X = \left[ (p \cdot k)^2 a_1 a_2^* a_2 a_1^* + k^4 (1 - a_1^* a_2^* a_2 a_1) \right] (1 - s_1 \cdot s_2) \]

\[ + p \cdot k k^2 (a_2 a_1^* + a_1 a_2^* + a_2 a_1 a_1^* + a_1 a_2 a_2^*) \cdot (s_1 s_2 - s_2 s_1) \]

\[ - k^4 (a_1 a_1^* + a_2 a_2^* + a_1 a_1 a_2 a_2^* + a_2 a_2 a_1 a_1^*) \cdot (s_1 s_2 + s_2 s_1) \]

\[ + \frac{2 \mu}{i} p \cdot k [K(s_1 + \sigma_2) (a_2 a_2^* a_1 a_1^* + a_1 a_1^* a_2 a_2^*)] \]

\[ + \frac{2 \mu}{i} k^2 [K(s_1 - \sigma_2) (a_1 a_1^* + a_2 a_2^* + a_1 a_1^* a_2 a_2^* + a_1 a_1^* a_2 a_1^*)]. \]
Here \( s_2 \) is another mirror-spin, defined by
\[
s_2 \equiv s_2 - (2/K^2) K \cdot s_2 ; \quad s_2 \cdot s_2 = -1 ;
\]
but \( p_2 \cdot s_2 = 0 ! \) \quad (5.7)

The scattering formula (5.6) is quite compact and looks simple, but it has two disadvantages: First the appearance of the mirror-spins instead of the original ones, second the special gauge of the photon polarizations; therefore the formula is neither gauge-independent nor reciprocity symmetric. All this disadvantage disappears if we, in the next section, introduce Stokes parameters and eliminate the mirror-spins.

\[ \text{§ 6. Introduction of Stokes Parameters} \]

We now adjust the scattering formula to arbitrary polarizations by introducing into Eq. (5.6) for \( \alpha \alpha^* \) an average by means of Eq. (3.15), and for \( s_1,s_2 \) the explicit expressions (4.7), (5.7). In all these expressions we get denominators \( 4 p_1 \cdot k_1 = p' \cdot k + k^2 \) and \( 4 p_1 \cdot k_2 = p' \cdot k - k^2 \). It is convenient, to eliminate all these denominators by help of the abbreviation
\[
y \equiv \frac{4 \mu^2 k^2}{(p' \cdot k)^2 - k^4} = \frac{\mu^2 k_1 \cdot k_2}{2 p_1 \cdot k_1 p_1 \cdot k_2} = \frac{\mu^2 k_1 \cdot k_2}{2 p_2 \cdot k_1 p_2 \cdot k_2}.
\]

(6.1)

In the rest frame of either electron 1 or electron 2, \( y \) is simply expressed by the scattering angle \( \theta \) of the photon:
\[
y = \frac{1 - \cos \theta}{2} \quad \text{in rest frame of } p_1 \text{ or } p_2 . \quad (6.2)
\]

We now express by \( y \) all the scalar products appearing in the algebra. First we mark that
\[
\frac{k^2(p \cdot \beta)^2}{(p' \cdot k)^2 - k^4} = 1 - y . \quad (6.3)
\]

We have now
\[
\beta_1 \cdot \beta_2 = 1 - 2 y ; \quad (6.4)
\]

\[
2 \mu^2 X = \left( 2 + \frac{k^2}{\mu^2 y} \right) \left( 1 + 2 y \right) \left( \eta_1 \eta_2 \sigma_1 \cdot \sigma_2 - \xi_1 \xi_2 \right) + \left( \eta_1 \eta_2 - \xi_1 \xi_2 \right) \sigma_1 \cdot \Phi \cdot \sigma_2 - \xi_1 \xi_2 \sigma_1 \cdot \sigma_2 + \frac{y^2}{\mu^2} \eta_1 \eta_2 \sigma_1 \cdot (k + p) \cdot (k - p) \cdot \sigma_2
\]

\[
- 4 y (1 - y) (1 - \xi_1) (1 - \xi_2) (1 - \sigma_1 \cdot \sigma_2) + 2 (1 - 2 y) (\zeta_1 \sigma_2 - \eta_1 \eta_2 - \zeta_1 \zeta_2) \sigma_1 \cdot \Phi \cdot \sigma_2 + \frac{y^2}{\mu^2} \xi_1 \xi_2 \sigma_1 \cdot (k + p) \cdot (k - p) \cdot \sigma_2
\]

\[
+ 2 \frac{y^2}{\mu^2} \xi_2 \sigma_1 \cdot (k + p) \cdot \sigma_2 + \frac{2 y^2}{\mu^2} \xi_1 \xi_2 \sigma_1 \cdot (p - k - p) \cdot \sigma_2 + \frac{2 y^2}{\mu^2} \eta_1 \eta_2 \sigma_1 \cdot (p + k) \cdot \sigma_2
\]

\[
+ \frac{y^2}{\mu^2} \det (K p \sigma_2 \sigma_1) \left( \eta_1 \xi_2 - \xi_1 \eta_2 \right) - \frac{y^2}{\mu^2 k^2} \det (K p k \cdot (\sigma_1 \sigma_2 - \sigma_2 \sigma_1)) \left( \eta_1 - \eta_2 \right)
\]

\[
s_1 \cdot s_2 = s_1 \cdot (I + \Phi) \cdot s_2 ;
\]

\[
\Phi = \frac{y}{2 \mu^2} [p p + k - (k p + p k) p' \cdot k / k^2] ;
\]

\[
s_1 \cdot \beta_1 = s_1 \cdot (y \beta + \chi) ; \quad s_2 \cdot \beta_2 = s_2 \cdot (y \beta + \chi) ;
\]

\[
s_1 \cdot \beta_2 = s_1 \cdot (y \beta - \chi) ; \quad s_2 \cdot \beta_1 = s_2 \cdot (y \beta - \chi) . \quad (6.6)
\]

The scalar products (6.5) and (6.6) exhibit the reciprocity symmetry according to Eq. (4.1) which in the definitions of the vectors themselves is missing. — In treating the two alternating product terms it is sufficient to calculate the \( \sigma_2 \) part; the \( \sigma_1 \) part then follows from reciprocity symmetry according to (4.1), where instead of \( \alpha_1 \longrightarrow \alpha_2 \) we now have \( \zeta_1 \rightarrow - \zeta_2 \); \( \eta_1 \rightarrow - \eta_2 \); \( \zeta_1 \rightarrow - \zeta_2 \) [sign change because of Eqs. (3.8), (3.12)].

In course of the algebra \( a \) is easily eliminated by means of (3.9), (3.10), (3.11). The vectors \( \beta, \chi \) then appear only in the form of the tensors \( \beta \beta, \chi \chi, \chi \beta + \beta \chi \), explicitly given by:
\[
(1 - y) \beta \beta = \frac{1}{2} \Phi + \frac{k k}{k^2} ,
\]

\[
\chi \chi = (1 - y) \left( \frac{1}{2} \Phi + \frac{p p}{k^2} \right) ,
\]

\[
p \cdot k \left( \frac{y}{2 \mu^2} \right) (\chi \beta + \beta \chi)
\]

\[
= - \left( 1 + \frac{k^2}{2 \mu^2 y} \right) \Phi - \frac{y}{2 \mu^2} (k + p \cdot p) .
\]

To simplify the terms containing determinants some use of Eq. (2.10) has to be made. — The final result in the form we consider most simple reads:
\[
-\frac{y^2}{\mu^2 k^2} \det \left( K k p \cdot (\sigma_1 \sigma_2 + \sigma_2 \sigma_1) \right) (\eta_1 + \eta_2)
\]
\[
+ \frac{y^2}{\mu^2 k^2} \det \left[ K k p \left( 1 - \frac{(p \cdot k)^2}{4 \mu^2 k^2} \right) \right] \cdot (\sigma_1 \sigma_2 + \sigma_2 \sigma_1) (\eta_1 \tilde{\xi}_2 + \tilde{\xi}_1 \eta_2)
\]
\[
+ 2 \frac{y^2}{\mu} p \cdot (\sigma_2 - \sigma_1) (\tilde{\xi}_1 + \tilde{\xi}_2) + \frac{2 y (1 - y)}{\mu} k \cdot (\sigma_2 + \sigma_1) (\tilde{\xi}_1 - \tilde{\xi}_2) + \frac{y^2}{2 \mu^3} (k \cdot p p - p^2 k)
\]
\[
\cdot (\sigma_2 + \sigma_1) (\tilde{\xi}_2 \tilde{z}_2 - \tilde{z}_2 \tilde{\xi}_1) - \frac{y^2}{2 \mu} (p \cdot k k + 4 \mu^2 p - \frac{(p \cdot k)^2}{k^2} p) \cdot (\sigma_2 - \sigma_1) (\tilde{\xi}_1 \tilde{z}_2 + \tilde{z}_1 \tilde{\xi}_2)
\]
\[
+ \frac{p \cdot k}{2 \mu^3 k^2} y^2 \det \left( K k p (\sigma_2 + \sigma_1) \right) (\eta_1 \tilde{\xi}_2 - \tilde{\xi}_1 \eta_2) + \frac{y^2}{2 \mu^3} \det \left( K k p (\sigma_2 - \sigma_1) \right) (\eta_1 \tilde{\xi}_2 + \tilde{\xi}_1 \eta_2).
\]

What concerns the physical meaning of the Stokes parameters herein, \( \tilde{\xi}_1, \eta_1, \tilde{\xi}_1 \) clearly describe the polarization state of the primary light. \( \xi_2, \eta_2, \tilde{\xi}_2 \) describe the polarization component, for which we ask; this is either a pure (linear, circular, elliptic) polarization, so \( \xi_2^2 + \eta_2^2 + \tilde{\xi}_2^2 = 1 \); or no specification of polarization, that is: polarization average, \( \xi_2^2 = \eta_2^2 = \tilde{\xi}_2^2 = 0 \) — in this case the majority of the terms vanishes. — Similarly we specify the spins \( \sigma_1, \sigma_2 \). Clearly \( \sigma_2 \) is the secondary spin direction, for which we ask, a space-like unit vector; if we are not interested in \( \sigma_2 \), then we average \( \sigma_2 = 0 \), which again makes most of the terms vanish. On the other hand, \( \sigma_1 \) indicates the spin (polarization) of the primary electrons; if they were fully polarized, then \( \sigma_1 \cdot \sigma_1 = -1 \); in reality they are at most partly polarized (ferromagnets), so we have to understand by \( \sigma_1 \) the average spin \( (0 \leq \sigma_1 \cdot \sigma_1 \leq 1) \); in case of unpolarized primary electrons \( \sigma_1 = 0 \).

This now is the formula for Compton-scattering. The annihilation formula only differs by a sign change of \( \tilde{\xi}_2 \) and by the definitions of \( p, k, K \) according to Eqs. (1.6), (3.1) and footnote 4.

The first two terms of Eq. (6.11) contain the Klein-Nishina formula. To compare with the usual form, mark that
\[
2 + \frac{k^2}{\mu^2} y = \frac{k_1 \cdot p_1}{k_2 \cdot p_1} + \frac{k_2 \cdot p_1}{k_1 \cdot p_1}.
\]

If \( p_1 \) (or \( p_2 \)) is at rest, this is \( \omega_1/\omega_2 + \omega_2/\omega_1 \). Furthermore because of Eq. (6.2) in either of the rest frames \( 4 y (1 - y) = \sin^2 \theta \).

§ 7. Check with Previous Formulas

To compare our formula with Fr\(^1\) and LT\(^2\) we have to introduce instead of the covariant spin-four-vectors \( \sigma \) space vectors, called \( \xi \) in LT and \( \sigma \) in Fr, indicating the average spin vector in the rest system of the resp. electron. \( \sigma \) is expressed by
\[
\sigma = \left( \frac{p \cdot \xi}{\mu}, \xi + \frac{p \cdot \xi}{\mu (p_0 + \mu)} \right),
\]
(\( p_0, p \) = \( p \) being the momentum vector of the electron. Specified to primary electron at rest (6.11) now checks completely with Fr\(^1\), and with LT, except a missing subscript 0 in LT (2.7) "\( k_0 \cos \theta + k \cos \theta \)". and a missing factor 2 of the last line of LT (2.16). Some of the LT formulas are considerably simplified when products of pseudoscalars are expressed by scalar products (Gram’s determinant):

\[
8 \Phi_2(\xi^0, \xi) = (1 + \cos^2 \theta) \xi \cdot \xi^0 + (1 - \cos \theta) (k \cdot \xi \xi^0 \cdot n_0 - k^0 \cdot \xi \xi^0 \cdot n)
\]
LT (2.11)
\[
8 \Phi_3(\xi, \xi^0, \zeta) = (1 - \cos \theta) \left\{ \xi_1 (1 + \cos \theta) \xi^0 \cdot n + n^0 \xi^0 \cdot \mathbf{K}^0 - \mathbf{K}^0 \cdot \xi^0 \cdot n \right. \\
+ (k^0 - \mathbf{K}) \cdot \xi^0 \left. + \frac{(k^0 - k \cos \theta)(k^0 \cdot \xi^0 \cdot n - (k^0 \cos \theta - k)\xi^0 \cdot \mathbf{n})}{k^0 - k + 2} \right) \right\} \\
+ \xi_2 \left\{ k^0 (\mathbf{n} - \mathbf{n}^0 \cos \theta) \cdot \xi \times \xi^0 + \frac{k^0 + k}{k^0 - k + 2} (k^0 - \mathbf{K}) \cdot \xi^0 \cdot \mathbf{n} \times \mathbf{n} \right. \\
- \xi_3 (1 + \cos \theta) (k^0 - \mathbf{K}) \cdot \xi \times \xi^0 \\
+ (\xi_1^0 \xi_2^0 - \xi_2^0 \xi_1^0) \frac{k^0}{k^0 - k + 2} (1 - \cos \theta) \left\{ \left( k^0 + \mathbf{K} \right) \cdot \xi^0 \cdot \mathbf{n} \times \mathbf{n} \right. \\
\left. + \frac{(k^0 + k)^2}{k^0 - k + 2} (k^0 - \mathbf{K}) \cdot \xi^0 \cdot (\mathbf{n} \times \mathbf{n}) \right\} \\
- (\xi_1^0 \xi_2^0 + \xi_2^0 \xi_1^0) \frac{k^0}{k^0 - k + 2} (1 - \cos \theta) \left\{ \mathbf{n} \cdot \xi^0 \cdot \mathbf{K} - \mathbf{n} \cdot \xi^0 \cdot \mathbf{K}^0 \right. \\
\left. + 2 (\mathbf{K} \cdot \xi^0 \cdot \mathbf{n}^0 - \mathbf{K} \cdot \xi^0 \cdot \mathbf{n}) \right\} \\
+ (\xi_1^0 \xi_2^0 + \xi_2^0 \xi_1^0) \left\{ \left( k^0 - \mathbf{K} \right) \cdot \xi^0 \cdot (\mathbf{n} \times \mathbf{n}) \right. \\
\left. + \xi_3 (1 + \cos \theta) (\mathbf{n} \times \mathbf{K} \cdot \xi^0 \cdot \mathbf{n} \times \mathbf{n}) \right\} \\
+ \frac{k^0 - k}{k^0 - k + 2} \left\{ (1 - \cos \theta)^2 \xi_1^0 \xi_2^0 + 2 \cos \theta (\xi_1^0 \xi_1^0 + \xi_1^0 \xi_2^0 + \xi_2^0 \xi_3^0) + 2 \xi_3^0 \xi_3^0 \right\}. 
\]

Landau and Lifschitz started calculating \( F_c \) [Eq. (1.2)] up to a certain extent; we carried through the completion of their idea, again achieving result (6.11).

Low made the computer calculate a part of the formula for Compton scattering, omitting all terms containing determinants; his result turns out to agree with ours.

2. F. W. Lipps and H. A. Tolhoek, Physica 20, 85, 395 (1954); quoted as LT.
4. Pair annihilation differs from Compton scattering just by the substitution \( p_2 \rightarrow -p_2, \alpha_2 \rightarrow -\alpha_2, \zeta_1 \rightarrow -\zeta_1, \zeta_1 \rightarrow -\zeta_1 \).