Electron-Electron Bremsstrahlung in a Hot Plasma

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Using the differential cross section for electron-electron (e-e) bremsstrahlung exact to lowest-order perturbation theory the bremsstrahlung spectra and the total rate of energy loss of a hot electron gas are calculated. The results are compared to the nonrelativistic and extreme-relativistic approximations available and to previous work. The e-e contribution to the bremsstrahlung emission of a hydrogenic plasma is found to be not negligible at temperatures above \( \approx 10^8 \) K, especially in the short-wavelength part of the spectrum.

1. Introduction

The observation of the X-radiation emanating from a hot Maxwellian plasma is very useful in obtaining quantitative estimates on the physical parameters such as electron temperature and density. The basic requirement for the deduction of these quantities is the knowledge of bremsstrahlung cross sections. Whereas there are no problems concerning the X-radiation arising from electron-ion (e-i) collisions, the cross section for electron-electron (e-e) bremsstrahlung was known only in nonrelativistic (quadrupole) and in extreme-relativistic approximations. In order to get the bremsstrahlung spectra in the intermediate energy region, one had to interpolate between the two limiting cases. Another difficulty is due to the fact that the cross section for e-e bremsstrahlung should be known for arbitrary directions of the electron velocities while it is usually given in a special frame of reference, e.g., the center-of-mass system or the rest system of one of the colliding particles. Now a covariant formula for the cross section of e-e bremsstrahlung differential with respect to photon energy and angle is available which is exact to lowest-order perturbation theory.

Using this expression it is possible to compute the e-e bremsstrahlung spectra of a hot plasma and to specify the range of validity and the accuracy of the approximation formulae. Additionally it makes it feasible to calculate the total energy emitted by an electron gas of temperature \( T \). This energy can exceed the total bremsstrahlung rate generated by electron-proton (e-p) collisions in a hydrogenic plasma at very high temperatures. The results are compared to those obtained by Maxon employing the interpolation method.

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2. Bremsstrahlung Spectra

The number of photons emitted per unit time, per unit volume, per unit energy interval by an electron gas of uniform number density \( n_e \) at temperature \( T \) is given by

\[
P_{\text{ee}}(k, \tau) = \frac{1}{2} \frac{n_e^2 c}{4\pi} \int \frac{d^3 p_1}{(2\pi)^3} \int \frac{d^3 p_2}{(2\pi)^3} f(p_1) f(p_2) \left( \frac{1}{(p_1 p_2)^2} - 1 \right) \frac{\sigma}{d\epsilon} \frac{\epsilon^2}{d\tau}
\]

where \( c \) is the velocity of light, \( \epsilon_1 \) and \( \epsilon_2 \) are the total electron energies, and \( \sigma/d\epsilon \) is the cross section for generating a photon of energy \( k \) in the collision of two electrons with the momenta \( p_1 \) and \( p_2 \); the energies \( \epsilon_1 \), \( \epsilon_2 \), and \( k \) are in units of the electron rest energy \( mc^2 \), the momenta \( p_1 \) and \( p_2 \) are in units of \( mc \), and \( (p_1 p_2) = \epsilon_1 \epsilon_2 - p_1 \cdot p_2 \) is the invariant product of the four-momenta \( (\epsilon_1, p_1) \) and \((\epsilon_2, p_2)\).

The relativistic Maxwell-Boltzmann equilibrium distribution of electrons at temperature \( T \) is described by

\[
f(p) = \frac{e^{-\epsilon/T}}{4\pi c K_2 (1/T)}
\]

where \( \tau = k_B T/mc^2 \) \((k_B = \text{Boltzmann constant})\) is the non-dimensional temperature and the function \( K_2 \) is the modified Bessel function of the second kind. \( f(p) \) is normalized to unity,

\[
\int f(p) d^3p = 1.
\]

In (2.1) the quantity \( (p_1 p_2)^2 - 1/\epsilon_1 \epsilon_2 \) originates from the invariant expression for the flux of incident particles \( \tau \), and the factor \( 1/2 \) prevents from the double counting of each electron.

The cross section \( \sigma/d\epsilon/d\tau \) depends, apart from \( \epsilon_1 \), \( \epsilon_2 \), and \( k \), only on the angle \( \phi \) between the vectors \( p_1 \) and \( p_2 \). Since the distribution \( f(p) \) is a function...
of the absolute value \( p = |p| = \sqrt{\varepsilon^2 - 1} \) of the electron momentum, most of the angle integrations in (2.1) can be performed easily. Replacing the integration variable \( \varepsilon \) by \( \mu \equiv (p_1 p_2) = \varepsilon_1 \varepsilon_2 - p_1 p_2 \cos \varepsilon \), one obtains

\[
P_{ee}(k,\tau) = \frac{e n_e^2}{2 \tau K_2(1/\tau)^2} \int d\varepsilon_1 \int d\varepsilon_2 \int d\mu \sqrt{\mu^2 - 1} \frac{d\alpha}{dk}.
\]

The integration intervals in (2.4) follow from the kinematics of the e-e process. All the combinations of the variables \( \varepsilon_1, \varepsilon_2, \) and \( \mu \) are allowed which satisfy the condition

\[
k \leq \varepsilon_1 + \varepsilon_2 - \sqrt{(\varepsilon_1 + \varepsilon_2)^2 - 2(\mu + 1)}.
\]

Since the computation of \( d\sigma/dk \) from the differential cross section \( d^2\sigma/dkd\Omega \) available requires two angle integrations in case of arbitrary directions of the momenta \( p_1 \) and \( p_2 \), one has to carry out a total of 5 integrations to obtain \( P_{ee}(k,\tau) \).

For “low” temperatures \( \tau \ll 1 \) an approximate formula for \( P_{ee}(k,\tau) \) can be derived. Fedusin and Garibyan have calculated the cross section for e-e quadrupole emission valid in the center-of-mass system of the colliding electrons for nonrelativistic energies \( \varepsilon_1 - 1 \ll 1, \varepsilon_2 - 1 \ll 1 \). This expression, generalized to arbitrary frames of reference, is given by

\[
\frac{d\sigma_{NR}}{dk} = \frac{4}{15} \frac{a r_0^2 c}{k} F\left(\frac{4 k}{(p_1 - p_2)^2}\right)
\]

where \( a = e^2/\hbar c \approx 1/137 \) is the fine-structure constant, \( r_0 = e^2/m c^2 \) is the classical radius of the electron, and the function \( F \) has the form

\[
F(x) = \left[17 - \frac{3 x^2}{(2-x)^2}\right] V(1-x) + \frac{12}{(2-x)^3} \left[7 x^2(2-x)^2 - 3 x^4\right] \ln \frac{1 + \sqrt{1-x}}{\sqrt{x}}.
\]

Using the ordinary Maxwell distribution

\[
N_{NR}(p) = (2\pi)^{-3/2} e^{-p^2/2}\tau
\]

Equation (2.1) reduces to

\[
P_{eeNR}(k,\tau) = \frac{3}{8} n_e^2 c (2\pi)^{-3} \int d^3 p_1 \int d^3 p_2 \left| p_1 - p_2 \right| \exp \left\{ - (p_1^2 + p_2^2)/2 \tau \right\} \frac{d\sigma_{NR}}{dk}.
\]

It is convenient to introduce the new variables \( p = p_1 - p_2 \) and \( P = p_1 + p_2 \). Then the integration over \( P \) can be carried out, and the result is

\[
P_{eeNR}(k,\tau) = \frac{8\pi}{15} a r_0^2 c n_e^2 (2\pi)^{-3/2} k \int_0^1 \frac{F(x)}{x^3} e^{-k(x)} dx.
\]

This formula has been derived by Maxon and Corman. The integral

\[
I(k/\tau) = \int_0^1 \frac{F(x)}{x^3} e^{-k(x)} dx
\]

can be computed numerically. The values for \( 0.025 \leq k/\tau \leq 5 \) are given in Table 1.

For temperatures up to \( T = 10^8 \) K the nonrelativistic formula (2.10) is a very good approximation to Equation (2.4). The values of \( P_{ee}^{NR}(k,\tau) \) are slightly too low throughout the spectrum. At \( k_B T = 10 \) keV corresponding to \( T = 1.16 \cdot 10^8 \) K the relative error is less than \( 1\% \) for the photon energies above 1 keV.

The dipole spectrum emitted by a nonrelativistic hydrogenic plasma at temperature \( T \) is given by

\[
P_{ee}^{NR}(k,\tau) = \frac{16}{3} a r_0^2 c n_e n_p \frac{r}{k} \int_{2/\tau}^1 \frac{2/\tau}{(\pi \tau)} e^{-k/2\tau} K_0(k/2 \tau)
\]

where \( n_p \) is the proton number density and \( K_0 \) the modified Bessel function of the second kind. At temperatures \( k_B T \geq 20 \) keV the contributions of higher multipoles to the bremsstrahlung spectrum become important. They are most simply included by employing the correction factor

\[
R = 1/(B - v k)
\]

derived by Quigg, who has given a table with the parameters \( B \) and \( v \) as a function of \( \tau \). Then

\[
P_{ep}(k,\tau) = R \cdot P_{ep}^{NR}(k,\tau).
\]

In the nonrelativistic approximation the ratio of e-e (quadrupole) to e-p (dipole) bremsstrahlung emis-
Table 1. Function $I(k/r) = \int_0^{\infty} F(z) e^{-k/(z^2)} dz$.

<table>
<thead>
<tr>
<th>$k/r$</th>
<th>0.025</th>
<th>0.050</th>
<th>0.075</th>
<th>0.10</th>
<th>0.20</th>
<th>0.30</th>
<th>0.40</th>
<th>0.5</th>
</tr>
</thead>
<tbody>
<tr>
<td>$I(k/r)$</td>
<td>1.311 $\times 10^{-3}$</td>
<td>2.910 $\times 10^{-4}$</td>
<td>1.193 $\times 10^{-4}$</td>
<td>6290</td>
<td>1299</td>
<td>497.8</td>
<td>245.9</td>
<td>139.5</td>
</tr>
<tr>
<td>$k/r$</td>
<td>0.6</td>
<td>0.7</td>
<td>0.8</td>
<td>0.9</td>
<td>1.0</td>
<td>1.2</td>
<td>1.4</td>
<td>1.6</td>
</tr>
<tr>
<td>$I(k/r)$</td>
<td>3.6</td>
<td>3.8</td>
<td>4.0</td>
<td>4.2</td>
<td>4.4</td>
<td>4.6</td>
<td>4.8</td>
<td>5.0</td>
</tr>
</tbody>
</table>

The photon energy $h\nu = mc^2 k = 60$ keV. The relative error made by using (2.10) increases from $\approx 8\%$ at $h\nu = 60$ keV to $\approx 30\%$ at $h\nu = 300$ keV. It is seen that the contribution of e-e bremsstrahlung to the emission of a hydrogenic plasma is no more negligible at such high temperatures, especially in the short-wavelength part of the spectrum. At $h\nu = 300$ keV, e.g., the e-e emission amounts to about 70\% of the e-p emission.

The values of $P_{ee}(k, \tau)$ obtained by Maxon with the aid of the interpolation method generally compare favourably to the present ones except for the points at $h\nu = 10$ keV, $h\nu = 20$ keV, and $h\nu = 200$ keV which differ by $-12\%$, $-11\%$, and $+16\%$, respectively. This is certainly due to his "eyeball fit" in doubly logarithmic diagrams. Therefore the interpolation method is not very reliable if one wants to have more than a rough estimate. Although Maxon's values of $P_{ep}(k, \tau)$ sometimes differ considerably, his results for $P_{EE}(k, \tau)$ agree very well with those calculated from $P_{EP}(k, \tau)$ by means of the correction factor (2.13).

In the preceding calculations screening effects have been ignored completely. For e-p bremsstrahlung this can be justified by computations made with Debye potentials. It appears that screening by free electrons is important only for extremely low frequencies. In a recent investigation on e-i bremsstrahlung in a Debye plasma Kvasnica has shown that screening effects are beginning to play a part if the quantity

$$\lambda = h/(2m^2 c^3 D^2 \tau)$$

(2.16)

becomes greater than about $10^{-3}$, where

$$D = \sqrt{\frac{m c^2 \tau}{4 \pi e^2 (n_e + Z^2 n_i)}}$$

(2.17)

is the Debye radius, $Z$ is the atomic number, and $n_i$ the ion density. In a hydrogenic plasma ($Z = 1$) of
density \( n_e = n_p = 10^{16} \text{ cm}^{-3} \), \( D \approx 3.8 \times 10^{-3} \sqrt{T} \text{ cm} \) and \( \lambda \approx 5.5 \times 10^{-17} / r^2 \); that is, the deviations from the Coulomb-field approximation become important only below \( T \approx 1400 \text{ K} \), temperatures which are much lower than those considered in the present paper.

These estimates, valid for e-e bremsstrahlung, are correct in order-of-magnitude for e-e bremsstrahlung as well.

### 3. Bremsstrahlung Rate of Energy Loss

The total energy emitted as e-e bremsstrahlung per unit space-time volume is given by

\[
W_{ee}(\tau) = m c^2 \int_0^\infty k P_{ee}(k, \tau) \, dk .
\]

(3.1)

Instead of integrating the photon spectrum \( P_{ee}(k, \tau) \), the total energy \( W_{ee}(\tau) \) can be calculated directly by using the cross section of e-e bremsstrahlung in the center-of-mass system (c.m.s.) \(^3,8\). Let all quantities in the c.m.s. be labelled by the suffix cm; then the energy of a photon in an arbitrary frame of reference S is

\[
k = \frac{k_{cm} + \mathbf{v}_{cm} \cdot \mathbf{k}_{cm}}{c} / V \sqrt{1 - v_{cm}^2 / c^2}
\]

(3.2)

where

\[
\mathbf{v}_{cm} = (p_1 + p_2) / (e_1 + e_2)
\]

(3.3)

is the velocity of the center-of-mass, and \( p_1 \), \( p_2 \), \( e_1 \), and \( e_2 \) are the momenta and energies, respectively, of the electrons in the system S. The total energy per unit space-time emitted in S by electrons with the momenta \( p_1 \) and \( p_2 \) is

\[
\int \int k \left( \frac{d^2 \sigma}{dk_{cm} d\Omega_{cm}} \right) d\Omega_{cm}.
\]

(3.4)

The upper limit of integration, \( k_{max} = p_{cm}^2 / \epsilon_{cm} \), can be expressed by the invariant product \( \mu = (p_1 p_2) / c^2 \):

\[
k_{max} = \frac{\mu - 1}{\sqrt{2} (\mu + 1)} .
\]

(3.8)

The integral

\[
Q_{cm} = \int \frac{d\sigma}{dk_{cm} d\Omega_{cm}}
\]

(3.9)

is dependent on the energy \( \epsilon_{cm} \) and the momentum \( p_{cm} \) of the electrons in the c.m.s. which, as a function of \( \mu \), have the form

\[
\epsilon_{cm} = V^{1/2} (\mu + 1), \quad p_{cm} = V^{1/2} (\mu - 1).
\]

(3.10)

Finally one has to average \( R_{ee}(p_1, p_2) \) over the Maxwell-Boltzmann distributions of the electrons to obtain

\[
W_{ee}(\tau) = \frac{1}{2} \int \int R_{ee}(p_1, p_2) f(p_1) f(p_2) d^3 p_1 d^3 p_2 = n_e^2 m c^3 \left[ 2 \pi K_2(1 / \tau) \right]^{-2} \int_1^\infty \int_1^\infty p_1 p_2 (e_1 + e_2) e^{-(e_1 + e_2) / \tau} \int d\xi \sqrt{\frac{2}{\mu - 1}} Q_{cm}(\mu)
\]

(3.11)
Since $Q_{\text{cm}}(\mu)$ has to be computed from the differential cross section, one needs again 5 numerical integrations to calculate $W_{\text{ee}}(\tau)$. It is, however, possible to employ an approximation procedure which requires only the 2 integrations over $\varepsilon_1$ and $\varepsilon_2$. For this purpose one has to find out an analytical expression for $Q_{\text{cm}}(\mu)$ which satisfies the condition that the integration over $\mu$ in (3.11) can be done in closed form. The procedure adopted for determining this function is to start from the nonrelativistic and the extreme-relativistic formulae for $Q_{\text{cm}}$ and then to establish an interpolation formula comprising these two limiting cases.

In the nonrelativistic approximations the cross section (2.6) can be used resulting in (henceforth the suffix cm of the variables $k$, $e$, and $p$ is omitted)

$$Q_{\text{cm}}^{\text{NR}} = \frac{4}{15} a r_0^2 \int F(k/p^2) \, dk = \frac{4}{15} a r_0^2 p^2 \int F(x) \, dx = \frac{20}{3} \left( \frac{11}{3} - \frac{\pi^2}{4} \right) a r_0^2 p^2 \approx 8.00 a r_0^2 p^2. \quad (3.12)$$

For very high energies the cross section has the form

$$\frac{d\sigma}{d\varepsilon} = \frac{8 a r_0^2}{\varepsilon k} \left[ \frac{4}{3} (\varepsilon - k) + \frac{k^2}{\varepsilon} \right] \left[ \ln \left( \frac{4 \varepsilon^2}{k (\varepsilon - k)} \right) - 1 \right], \quad (3.13)$$

resulting in

$$Q_{\text{cm}} = 16 a r_0^2 e \left[ \ln (2 \varepsilon) - \frac{1}{2} \right]. \quad (3.14)$$

An interpolation formula meeting the above conditions is given by

$$Q_{\text{cm}} \approx 8 a r_0^2 \frac{p^2}{\varepsilon} \left[ 1 - \frac{4 p}{3 \varepsilon} + \frac{2}{3} \left( 2 + \frac{p^2}{\varepsilon^2} \right) \ln (\varepsilon + p) \right]. \quad (3.15)$$

The error made by using the approximation (3.15) is less than 1% for the kinetic electron energies $E_{\text{cm}} < 50$ keV and $E_{\text{cm}} > 5$ MeV and is never greater than 6%.

If the quantities $\varepsilon$ and $p$ in (3.15) are expressed by the variable $\mu$ according to (3.10) the integral

$$J(\varepsilon_1, \varepsilon_2) = \int \frac{\sqrt{2} (\mu - 1) Q_{\text{cm}}(\mu) \, d\mu}{e^{\varepsilon_1 - \varepsilon_2}} \quad (3.16)$$

can be performed analytically:

$$J(\varepsilon_1, \varepsilon_2) = 4 a r_0^2 \left[ \frac{\mu - 2}{2} \sqrt{\mu^2 - 1} - \frac{11}{12} \mu^2 + \frac{20}{3} \mu - \frac{8}{3} \ln (\mu + 1) + \left\{ \frac{3}{2} + \frac{\mu - 8}{3} \mu + 1 \right\} \ln (\mu + \sqrt{\mu^2 - 1}) + \frac{7}{4} \left\{ \ln (\mu + \sqrt{\mu^2 - 1}) \right\}^2 \right] e^{\varepsilon_1 - \varepsilon_2}. \quad (3.17)$$

For $x = \mu - 1 \ll 1$ it is expedient to employ in numerical computations the form

$$J(\varepsilon_1, \varepsilon_2) \approx a r_0^2 \left[ 2 x^2 \sqrt{2 x} \left( \frac{4}{5} - \frac{x}{7} \right) + \frac{5}{18} x^4 \right] e^{\varepsilon_1 - \varepsilon_2}. \quad (3.18)$$

where the relative error compared to (3.17) is less than $5 \cdot 10^{-4}$ for $x < 0.2$.

By means of the function $J(\varepsilon_1, \varepsilon_2)$ the energy loss of a plasma due to e-e bremsstrahlung can be evaluated easily. It follows from (3.11)

$$W_{\text{ee}}(\tau) = m e^3 n_e^2 \left[ 2 \tau e^{1/\tau} K_2(1/\tau) \right]^{-2} \int dx_1 \int dx_2 (\varepsilon_1 + \varepsilon_2) e^{-(\varepsilon_1 + \varepsilon_2 - 2)/\tau} J(\varepsilon_1, \varepsilon_2). \quad (3.19)$$

Utilizing the symmetry $J(\varepsilon_1, \varepsilon_2) = J(\varepsilon_2, \varepsilon_1)$ and introducing the new variables $x_1 = \varepsilon_1 - 1$ and $x_2 = \varepsilon_2 - 1$, $W_{\text{ee}}(\tau)$ can be written

$$W_{\text{ee}}(\tau) = 2 m e^3 n_e^2 \left[ 2 \tau e^{1/\tau} K_2(1/\tau) \right]^{-2} \int dx_1 \int dx_2 (x_1 + x_2 + 2) e^{-x_1 - x_2/\tau} J(x_1, x_2). \quad (3.20)$$

In the nonrelativistic limit, $\tau \ll 1$, the total energy emitted is easily derived from Equation (2.10). If the order of integrations in

$$W_{\text{ee}}^{\text{NR}}(\tau) = m e^3 \int k P_{\text{ee}}^{\text{NR}}(k, \tau) \, dk = \frac{8 \pi}{15} a r_0^2 m e^3 n_e^2 (\pi \tau)^{-3/2} \int_0^{\infty} dk \, k^2 \int_0^{1} \, dx \frac{F(x)}{x^3} e^{-k/(x)}$$

For $p \ll 1$ it reduces to (3.12) up to terms of order $p^2$, and for $\varepsilon \gg 1$ it agrees with (3.14) through second order in $1/\varepsilon$. Together the expression (3.15) yields good results at intermediate energies. This follows from a comparison with the exact values of $Q_{\text{cm}}$ calculated by numerical integrations of the differential cross section. The error made by using the approximation (3.15) is less than 1% for the kinetic electron energies $E_{\text{cm}} < 50$ keV and $E_{\text{cm}} > 5$ MeV and is never greater than 6%.
is changed, one obtains $3^{12}$:

$$W_{ee}^{NR}(\tau) = \frac{16}{15} \pi a r_0^2 m c^3 n_e^2 (\tau/\tau)^{3/2} \int_0^1 F(x) \, dx = \frac{20}{9} (44 - 3 \pi^2) a r_0^2 m c^3 n_e^2 \tau \sqrt{\frac{\tau}{\tau}}$$

(3.21)

$$\approx 32.0 a r_0^2 m c^3 n_e^2 \tau \sqrt{\frac{\tau}{\tau}}$$

For extreme-relativistic energies, $\tau \gg 1$, the formula (3.14) can be employed to get

$$I_1(\epsilon_1, \epsilon_2) = 8 a r_0^2 \epsilon_1^2 \epsilon_2^2 \left[ \ln (4 \epsilon_1 \epsilon_2) - \frac{3}{2} \right]$$

(3.22)

and

$$W_{ee}(\tau) = 24 a r_0^2 m c^3 n_e^2 \tau \left[ \ln (2 \tau) - C + \frac{1}{2} \right]$$

(3.23)

where $C \approx 0.5772$ is Euler's constant. The latter expression was derived by Alexanian.$^8$

The total energy emitted as e-p bremsstrahlung per unit space-time volume in a hydrogenic plasma is obtained from

$$W_{ep}(\tau) = n_e n_p m c^3 \int d^3 p \int d\sigma_{ep}/d\epsilon$$

(3.24)

where $\epsilon$ and $p$ are the energy and momentum, respectively, of the electron, and $d\sigma_{ep}/d\epsilon$ is the e-p bremsstrahlung cross section. Substituting the distribution (2.2) into (3.24) and performing the angle integrations results in

$$W_{ep}(\tau) = n_e n_p m c^3 \frac{1}{\tau e^{1/2} K_2(1/\tau)}$$

(3.25)

$$\int_0^1 d\epsilon (\epsilon^2 - 1) e^{-(\epsilon - 1)/\tau} \int_0^{\epsilon - 1} dk \frac{d\sigma_{ep}}{dk}$$

In Born approximation the integral

$$\Phi(\epsilon) = \int_0^{\epsilon - 1} dk \frac{d\sigma_{ep}}{dk}$$

(3.26)

can be expressed in closed form$^{18}$. But the second integration over $\epsilon$ cannot be carried out analytically. Replacing $\Phi(\epsilon)$ by an approximate function which nowhere differs by more than 0.5% from $\Phi(\epsilon)$, Stickforth$^3$ has derived a formula for $W_{ep}(\tau)$,

$$W_{ep}(\tau) = \frac{8}{3} a r_0^2 m c^3 n_e n_p \frac{J_1(\tau) + J_2(\tau)}{\tau e^{1/2} K_2(1/\tau)}$$

(3.27)

where the functions $J_1(\tau)$ and $J_2(\tau)$ are given as series expansions.

Figure 2 shows in a doubly logarithmic scale the energy loss of a hydrogen plasma by e-e and e-p bremsstrahlung as a function of the temperature, calculated according to the formulae (3.20) and (3.27). Additionally the nonrelativistic and extreme-relativistic approximations of $W_{ee}(\tau)$ are plotted. At low temperatures $\tau \lesssim 10^{-3}$ the e-e rate is negligible compared to the e-p emission. The e-e contribution increases rapidly with temperature until the two curves cross at $\tau \approx 0.52$ or $k_B T \approx 264$ keV. In the extreme-relativistic energy region, $\tau \gg 1$, the e-p rate is given by $3^{12}$:

$$W_{ep}(\tau) = 12 a r_0^2 m c^3 n_e n_p \tau \left[ \ln (2 \tau) + \frac{3}{2} - C \right],$$

(3.28)

that is half of the e-e emission (3.23).

A comparison between the results of Eq. (3.20) and the nonrelativistic approximation (3.21) shows very good agreement (within 1%) for $\tau \lesssim 0.025$ or $T \lesssim 1.5 \times 10^8$ K. Above this temperature $W_{ee}^{NR}(\tau)$ becomes too low as can be expected from the behaviour of the spectrum $P_{ee}(k, \tau)$. In the extreme-relativistic limit the formula (3.23) is surprisingly correct down to $\tau \approx 1$ though it was derived under the assumption $\tau \gg 1$.

It is not easy to draw the values of $W_{ee}(\tau)$ with fairly good accuracy from the doubly logarithmic diagram obtained by Maxon$^4$ with the aid of the interpolation method. But yet it seems that they agree quite well with the present results. On the other hand the temperature of Maxon's crossing-
point, \( k_B T \approx 350 \text{ keV} \), is somewhat higher than in Figure 2.

4. Conclusions

In the calculation of X-ray spectra from hot hydrogenic plasmas the contribution of e-e bremsstrahlung becomes significant above \( k_B T \approx 10 \text{ keV} \), especially at high photon energies. Temperatures of this order of magnitude are assumed to occur in cosmic X-ray sources such as Cyg X-1 \(^{19}\) \((k_B T \approx 30 \text{ keV})\). Besides it has been suggested \(^{20}\) that thermal bremsstrahlung may be a source of \(\gamma\)-ray background flux which would require temperatures up to some MeV.

By comparison with the exact results the non-relativistic formula for the e-e spectrum has been found to be a very good approximation up to \( k_B T = 10 \text{ keV} \) so that it is sufficient in most practical cases. At temperatures \( k_B T = 100 \text{ keV} \) the non-relativistic expression becomes poorer mainly in the short-wavelength region. The total rate of energy loss by e-e bremsstrahlung in a hydrogenic plasma, \( W_{ee}(\tau) \), is negligible at temperatures below \( T \approx 5 \times 10^7 \text{ K} \). At \( T \approx 3 \times 10^8 \text{ K} \) the e-e emission is equal to the e-p rate and becomes twice the e-p rate at extreme-relativistic energies. The relativistic limit for \( W_{ee}(\tau) \) is correct down to \( \tau \approx 1 \). This fact may be utilized if the e-e rate is obtained by means of interpolation.

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