Forces on Small Particles in Nonhomogeneous Polyatomic Gases

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(Z. Naturforsch. 29 a, 1244—1252 [1974]; received March 15th, 1974)

Waldmann's work on forces on small particles suspended in a nonhomogeneous monatomic gas is extended to a polyatomic gas of rotating molecules. Under the assumption that the coefficient of diffuse reflection is equal to 1, the force the gas exerts on a particle is calculated from an expansion of the distribution function. In general, this force consists of three contributions which are determined by the flow velocity, the translational heat flux, and the pressure tensor in the gas. The latter force vanishes for spheroidal particles, leaving only the frictional and the thermal force. These two forces are given for prolate and oblative spheroids. In particular, the dependence of the thermal force on the orientation of the particle's axis relative to the temperature gradient is discussed, and a comparison with the hydrodynamic and free molecular results is made.

Particles at rest in a streaming and heat conducting gas are subject to forces which are proportional to the flow velocity and the heat flux in the gas, viz. the frictional and the thermal forces, respectively. These forces depend strongly on the ratios of the characteristic lengths of the system, i.e. the size a of the particle, the mean free path l of a gas molecule, and the size D of the macroscopic container. Here, the discussion is restricted to particles which are small on the macroscopic scale, i.e. where a < D applies.

With increasing mean free path the frictional force decreases from its pressure independent hydrodynamic value (for l < a) to the free molecular flow limit (for l ≫ a), which is proportional to pressure p. For small mean free paths l the frictional force is linear in a, for large values of l it is proportional to the surface of the particle, i.e. quadratic in a. The most prominent expressions for these limits are the formulas for spheres given by Stokes (for l ≪ a, see e.g. Lamb 1) and by Epstein 2 (for l ≫ a). The frictional force for a general ellipsoid is stated in the hydrodynamic limit by Lamb 1.

The thermal force in the hydrodynamic regime is caused by the thermal slip, hence it is proportional to the mean free path l, and increases with decreasing pressure like p⁻¹. In the free molecular flow limit (a ≪ D ≪ l) the thermal force decreases proportional to pressure. And in between (i.e. for a ≪ l ≪ D) it reaches its highest pressure independent value, first calculated roughly by Einstein 3, and more exactly by Waldmann 4, 5. Most calculations have been performed for spheres, namely by Hettner 6 and Epstein 7 (l ≪ a ≪ D), by Einstein 3 and Waldmann 4, 5 (a ≪ l ≪ D), and by Brock 8 and Phillips 9 (a ≪ D ≪ l). Expressions for disks in the hydrodynamic and the free molecular flow limits are due to Einstein 7 and Knudsen 10, respectively. A hydrodynamic calculation for a prolate spheroid 11 has been performed by Sexl 11, similar to Lamb's treatment of the frictional force for a general ellipsoid. For small mean free paths (l < a) the thermal force is a linear function of the particle size a, if l is large compared to a (i.e. for a < l < D, and a < D ≪ l) it is proportional to the surface of the particle.

All calculations for the intermediate regions are rather difficult, e.g. in the slip flow regime (l < a up to l ≲ a) elaborate boundary conditions have to be used. For a comprehensive review of theoretical and experimental work on the thermal force the article by Waldmann and Schmitt 12 may be consulted. More recent work in the slip flow regime is due to Jacobsen and Brock 13, Dwyer 14, and to Vestner, Kübel and Waldmann 15. The thermal force on a small sphere (radius a) between parallel plates (distance D) has recently been calculated by Phillips 9 for a ≲ D from a ≳ l up to D ≳ l using a two-stream Chapman-Enskog distribution function.

The purpose of this paper is the calculation of the forces on small spheroidal particles suspended in a polyatomic gas of rotating molecules. Specifically a ≲ l applies; in addition, for the thermal force, the validity of a ≲ l ≲ D is assumed, where D is the...
distance between parallel plates creating the temperature difference in the gas. In Waldmann's paper 4,5 a monatomic gas is considered, and the final expressions are evaluated only for spherical particles. The first part of this paper deals with the calculation of the force for general bodies, with the following assumptions:

1. The presence of the small particle does not disturb the distribution function of the approaching molecules.
2. All molecules are reflected diffusely from the surface, i.e., the coefficient of diffuse reflection is put equal to 1, whereas Waldmann 4,5 (for monatomic gases) allows values of \( z \) different from 1.

Then the force is calculated from a special expansion of the distribution function within the gas. For many polyatomic gases this expansion provides a good approximation for the treatment of flow and heat conduction in the presence of a magnetic field, i.e., for the Senftleben-Beenakker effect 16. The total force on the particle is determined by only three of the nine expansion coefficients, namely by the flow velocity \( \mathbf{v} \), the translational heat flux \( q_{\text{trans}} \) and the symmetric traceless part \( \mathbf{p} \) of the pressure tensor. This last type of force is only present if \( \mathbf{p} \) is non-zero, and if unsymmetric bodies are involved, like a hemisphere. For spheroidal particles only the frictional and the thermal forces remain. Explicit expressions for prolate and oblate spheroids are given.

The magnetic field dependent part of the translational heat flux gives rise to the thermomagnetic force, which has been calculated previously for spheres by Hess 17. Measurements of this magnetic field induced change in the thermal force on disks have been performed by Adair and coworkers 18. The thermomagnetic force for spheroids will be discussed in detail in a subsequent paper.

The second part of this paper is devoted to the discussion of the frictional and thermal forces for zero magnetic field. Special attention is given to flat disks and long rods, to the dependence of the thermal force on the orientation of these particles relative to the temperature gradient, and to the comparison with the corresponding results in the hydrodynamic and free molecular flow limits.

Finally, in the Appendix, the hydrodynamic expressions for the frictional 1 and for the thermal force (partly due to Sexl 11) are given, and the explicit formulas for prolate and oblate spheroids (see e.g., Ref. 19) are stated.

I. Calculation of the Force

1. A General Formula

A small body with a surface \( \sigma \) is immersed in a polyatomic gas of rotating molecules. The force on the surface element \( d\sigma \) with the outer unit normal \( \mathbf{n} \) is given by the momentum transfer from the gas:

\[
dK = - [\frac{1}{m} \mathbf{c} \cdot \mathbf{n} \cdot f_j^+ \, d^3c] + \frac{1}{m} \mathbf{c} \cdot \mathbf{n} \cdot f_j^-] \, d\sigma.
\]

(1.1)

Here, \( \mathbf{c} \) is the molecular velocity, \( c_n = \mathbf{c} \cdot \mathbf{n} \) its normal component, and \( m \) is the molecular mass. The distribution function for molecules in the rotational state \( j \) is given by the directional transfer from the gas:

\[
f_j = \frac{1}{\pi} \frac{m^2}{2kT} \exp\left(-\frac{E_j^r}{kT}\right).
\]

(1.2)

For \( f_j^+ \) the assumption is made, that all molecules are reflected diffusely with a Maxwell distribution

\[
f_j^+ = f_j^M(n^+, T^+_t, T^+_r).
\]

(1.3)

Since the coefficient \( z \) of diffuse reflection is put equal to 1, molecules are reflected with random orientation of their angular momenta. Incomplete accommodation of translational and rotational energy is taken into account by allowing \( T^+_t \) and \( T^+_r \) to be different from the temperature of the particle. All three temperatures are assumed to be constant over the surface \( \sigma \) of the particle. In

\[
f_j^M(n^+, T^+_t, T^+_r) = n^+ \left(\frac{m^2}{2\pi k T^+_r}\right)^{3/2} \exp\left\{-\frac{E_j^r}{k T^+_r}\right\} [1 - \left(\frac{m^2}{2k T^+_r}\right)^{3/2} \exp\left(-\frac{E_j^r}{k T^+_r}\right)]^{3/2},
\]

(1.4)

\( k \) is the Boltzmann constant, \( E_j = \hbar^2 j(j + 1)/2 \Theta \) is the rotational energy of a molecule (in the state \( j \)) with the moment of inertia \( \Theta \), and

\[
Q_j = \sum_j (2j + 1) \exp\left\{-E_j^r/k T^+_r\right\}
\]

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is the rotational partition sum. The density \( n^+ \) has to be determined such that no molecules accumulate at the surface \(^4\):

\[
\text{Tr} \int c_n f_j^+ \, d^3c + \text{Tr} \int c_n f_j^- \, d^3c = 0. \tag{1.5}
\]

If \( \Phi_j \) denotes the deviation of \( f_j \) from its equilibrium value \( f_j^0 = f_j^N(n_0, T_0, T_0) \) we have

\[
f_j = f_j^0 (1 + \Phi_j). \tag{1.6}
\]

Then, by use of Eqs. (1.2), (1.3), and of the relation

\[
\text{Tr} \int c_n f_j^N(n^+, T_{t^+}, T_{t^+}) = \pm \frac{1}{2} n^+ V T_0 T_{t^+} + X, \tag{1.7'}
\]

Eq. (1.5) can be solved for \( n^+ \):

\[
n^+ = n^0 (V T_0 T_{t^+} + X). \tag{1.7}
\]

Here, \( X \) is given by

\[
X = -2 \sqrt{\pi} \, \text{Tr} \int \hat{c} \, \mathbf{c}_n (f_j^0/n_0) \Phi_j \, d^3c, \tag{1.8}
\]

with \( \hat{c} = m/2k T_0 \). For small deviations from equilibrium Eq. (1.7) can be replaced by

\[
n^+ = n^0 (V T_0 T_{t^+} + X). \tag{1.7''}
\]

The temperatures \( T_{t^+} \) and \( T_{t^+} \) depend on the temperature of the particle and on the state of the gas. They are not calculated here since they do not contribute to the force

\[
\mathbf{K} = -\frac{1}{2} p_0 \int \int d\mathbf{n} (\mathbf{X} + \mathbf{n} \mathbf{X}). \tag{1.9}
\]

By \( p_0 = n_0 k T_0 \) the equilibrium gas pressure is denoted, and \( \mathbf{X} \) is an abbreviation for

\[
\mathbf{X} = 4 \text{Tr} \int \hat{c} \, \mathbf{c}_n (f_j^0/n_0) \Phi_j \, d^3c. \tag{1.10}
\]

In deriving Eq. (1.9) from Eqs. (1.1) – (1.3) has been made of the relation

\[
\text{Tr} \int \mathbf{m} \, \mathbf{c}_n f_j^N (n^+, T_{t^+}, T_{t^+}) d^3c = \frac{1}{2} n^+ k T_{t^+} \mathbf{n},
\]

of Eq. (1.7') and of

\[
\int \int d\mathbf{n} = 0.
\]

The general formula Eq. (1.9) can only be evaluated if the quantities \( \mathbf{X} \) and \( \mathbf{X} \) are known, i.e. if the non-equilibrium state of the gas, as described by \( \Phi_j \), is specified.

2. A Special Expansion of the Distribution Function

By application of the moment method \(^9\), a system of coupled differential equations for mean values can be derived from the generalized Boltzmann equation due to Waldmann and Snider \(^{21}\). To this end the distribution function is expanded into a complete set of symmetric traceless tensors \( \Phi_{\mu_1...\mu_\ell} \) built up from the dimensionless molecular velocity \( C = \beta_0^{1/2} \mathbf{c} \) and the rotational angular momentum \( \mathbf{J} \). Here this expansion is restricted to the finite set of tensors which has proven to be sufficient for a description of the magnetic field dependent flow and heat conduction in most polyatomic gases, i.e. for the Senftleben-Beenakker effect \(^{16}\):

\[
\Phi_j = P^j \Phi_j, \quad \Phi_j = \sum_{k=0}^\ell \sum_{\mu_1...\mu_\ell} \Phi_{\mu_1...\mu_\ell} \mathbf{a}_{\mu_1...\mu_\ell} + \Psi_{\ell, \alpha} b_{\ell, \alpha},
\]

\[
N_0 = N_1 = 3, \quad N_2 = 2. \tag{2.1}
\]

\( P^j \) is a projection operator on the \( j \)-subspace \(^{20}\). Differential equations for the expansion coefficients \( \Phi_{\mu_1...\mu_\ell} \) are obtained \(^{20}\) by taking moments of the Waldmann-Snider equation. The moments occurring in Eq. (2.1) are related to the following physical quantities: the density \( n \), the translational and rotational temperatures \( T_{\text{trans}} \) and \( T_{\text{rot}} \), the corresponding heat fluxes \( \mathbf{q}_{\text{trans}} \) and \( \mathbf{q}_{\text{rot}} \), the flow velocity \( \mathbf{v} \), the frictional pressure tensor \( \mathbf{p} \), the tensor polarization \( \mathbf{a} \) and the flux of tensor polarization \( \mathbf{b} \). The two quantities \( \mathbf{a} \) and \( \mathbf{b} \) are polarizations set up by collisions. They depend on a magnetic field, and due to the collisional coupling with \( \mathbf{p} \), \( \mathbf{q}_{\text{trans}} \) and \( \mathbf{q}_{\text{rot}} \) they account for the field effects of viscosity and thermal conductivity \(^{16}\).

Due to the relations

\[
\text{Tr} (\mathbf{v} - (\mathbf{v}))_0 \exp \{ - \epsilon \} = 0, \quad \text{Tr} \mathbf{J} \mathbf{J} \exp \{ - \epsilon \} = 0
\]

the tensors

\[
\Phi^{(3)} \sim -\epsilon, \quad \Phi^{(3)} \sim C (\epsilon - (\epsilon)_0),
\]

\[
\Phi^{(2)} \sim -\mathbf{J} \mathbf{J}, \quad \Psi \sim C \Phi^{(2)}
\]

give no contribution to the force; \( \epsilon = h^2 \mathbf{J}^2/2 \Theta k T_0 \), \((\ldots)_0\) refers to an equilibrium average. The tensors and moments relevant for the quantities \( \mathbf{X} \) and \( \mathbf{X} \) are then:

\[
\Phi^{(1)} = 1, \quad \mathbf{a}^{(1)} = (n - n_0)/n_0, \tag{2.2}
\]

\[
\Phi^{(2)} = \sqrt{\frac{1}{\mathbf{C}^2}} \left( \mathbf{C}^2 - \frac{1}{2} \right), \quad \mathbf{a}^{(2)} = \sqrt{\frac{3}{2}} \frac{T_{\text{trans}} - T_0}{T_0}, \tag{2.3}
\]

\[
\Phi^{(1)} = \sqrt{2} \mathbf{C}, \quad \mathbf{a}^{(1)} = \sqrt{2} \beta_0^{1/2} \mathbf{v}, \tag{2.4}
\]

\[
\Phi^{(2)} = \sqrt{\frac{2}{5}} C (\mathbf{C}^2 - \frac{3}{2}), \quad \mathbf{a}^{(2)} = \sqrt{\frac{9}{5}} \beta_0^{1/2} \mathbf{p}^{-1} \mathbf{q}_{\text{trans}}, \tag{2.5}
\]

\[
\Phi^{(1)} = \sqrt{2} \mathbf{C} \mathbf{C}, \quad \mathbf{a}^{(1)} = (1/2 \mathbf{p}^{-1} \mathbf{q}_{\text{trans}}) \mathbf{p}. \tag{2.6}
\]

The symbol \( \ldots \) indicates the symmetric traceless part of a tensor.
3. The Force

The contributions of these five tensors to \(X\) and \(X\) in Eqs. (1.8), (1.10) can easily be calculated, see e.g. the Appendix of Ref. 5 for the half-space integrals:

\[
X = \frac{n - n_0}{n_0} + \frac{1}{c} \frac{T_{\text{trans}} - T_0}{T_0}
- \frac{2}{c} \frac{n \cdot v + (2p_0)^{-1} n \cdot \bar{p}}{
+ \frac{1}{c} \frac{n \cdot v - \frac{1}{4p_0} n(n \cdot \bar{p})}{n_0}
\frac{2}{\pi c} \frac{(n + \delta) \cdot \left(v + \frac{1}{5p_0} q_{\text{trans}}\right)}{n_0}
+ \frac{n}{c} \frac{n \cdot v - \frac{1}{4p_0} n(n \cdot \bar{p})}{n_0}
\frac{2}{\pi c} \frac{(n + \delta) \cdot \left(v + \frac{1}{5p_0} q_{\text{trans}}\right)}{n_0}
\right]
\] (3.1)

Here \(\delta\) is the unit tensor, and \(c = \sqrt{8 k T_0 / \pi m}\) is the mean molecular velocity. By integration over the surface, cf. Eq. (1.9), all terms linear in the normal \(n\) vanish, thus the force

\[
K = p_0 \int d\sigma \left[ \frac{2}{\pi c} \left( n n + \delta \right) \cdot \left( v + \frac{1}{5p_0} q_{\text{trans}} \right) + \frac{1}{c} \frac{n \cdot v - \frac{1}{4p_0} n(n \cdot \bar{p})}{n_0} \right]
\] (3.3)

is determined by only three moments, namely the flow velocity \(v\), the translational heat flux \(q_{\text{trans}}\), and the pressure tensor \(\bar{p}\).

The integrals \(\int d\sigma \frac{n n}{n_0}\) and \(\int d\sigma n nn\) are characteristic for the surface of the particle. Whereas the second rank tensor is nonzero for all types of bodies (its trace gives the surface area), the third rank tensor vanishes for highly symmetric bodies, e.g. for spheroids. For a circular cylinder of length \(L\) and radius \(a\) with one spherical cap (i.e. a hemisphere for \(L = 0\)) one has:

\[
\int d\sigma = 2 \pi a L + 3 \pi a^2,
\int d\sigma n nn = \delta (\pi a L + \frac{3}{2} \pi a^2) + s s (\pi a^2 - \pi a L),
\int d\sigma n\mu n\nu n_1 = \frac{1}{2} \pi a^2 \left[ s\mu s\nu + s\nu s\mu + s\mu s\nu - 5 s\mu s\nu s\nu \right];
\]

\(s\) is the unit vector parallel to the cylinder axis pointing towards the circular cap. If such a particle is placed in a plane Couette flow the existence of the pressure tensor \(\bar{p}\) leads to small corrections in the frictional force of the order \(l/D\) (\(D\) is the plate spacing). A method for creating a nonzero value of \(\bar{p}\) in a stagnant polyatomic gas has been pointed out by Hess.\textsuperscript{22} If a tensor polarization is maintained externally (e.g. by optical pumping), a pressure tensor \(\bar{p}\) is created due to collisions. This could give rise to a force on a nonsymmetric body.

For spheroidal particles, however, the total force

\[
K = K^v + K^t
\] (3.4)

consists only of the frictional force

\[
K^v = k^v s s \cdot \bar{v} + k^v (\delta - s s) \cdot \bar{v}, \bar{v} = (1/\varepsilon) \bar{v},
\]

\[
k^v = p_0 \frac{F}{2} (1 + (1 + \varepsilon/2) - 2 N),
\]

\[
k^t = p_0 \frac{F}{2} (1 + (1 + \varepsilon/2) N),
\]

and of the thermal force

\[
K^t = \int s s \cdot \bar{Q} + \int (\delta - s s) \cdot \bar{Q}, \bar{Q} = (1/\varepsilon) q_{\text{trans}},
\]

\[
\int = -F (4/5 \varepsilon) (1 - N), \int = F (2/5 \varepsilon) (1 + N).
\]

Here, \(s\) denotes a unit vector parallel to the symmetry axis, \(F\) is the surface area of the particle, and \(N\) is defined by

\[
(1/F) \int d\sigma \frac{n n}{} = N \delta + (1 - 3 N) s s,
\] (3.9)

i.e. \(1 - 3 N\) characterizes the deviation of the particle from the sphere. If \(a_{\parallel}\) and \(a_{\perp}\) denote the semi-axis parallel and perpendicular to \(s\) respectively, the quantities \(F\) and \(N\) can be written as

\[
F = 2 \pi a_{\perp}^2 + 2 \pi a_{\parallel}^2 \frac{1 + \varepsilon}{1 - \varepsilon},
\]

\[
N = \frac{1 - \varepsilon^2}{2 \varepsilon} \left[ \frac{1 + \varepsilon}{1 - \varepsilon} \right] \left[ \left[ \frac{1 - \varepsilon^2}{2 \varepsilon} \frac{1 + \varepsilon}{1 - \varepsilon} + 1 \right]^{-1},
\]

with \(\varepsilon = (1 - a_{\parallel}^2 / a_{\perp}^2)^{1/2}\) for oblate spheroids, and as

\[
F = 2 \pi a_{\perp}^2 + 2 \pi a_{\parallel}^2 \frac{1 + \varepsilon}{1 - \varepsilon},
\]

\[
N = \frac{1}{2 \varepsilon} \left[ \sqrt{1 - \varepsilon^2} - \frac{1 - 2 \varepsilon^2}{\varepsilon} \arcsin \varepsilon \right] \left[ \sqrt{1 - \varepsilon^2} + (1/\varepsilon) \arcsin \varepsilon \right]^{-1},
\] (3.11)

with \(\varepsilon = (1 - a_{\parallel}^2 / a_{\perp}^2)^{1/2}\) for prolate spheroids.

An inspection of Eqs. (3.6), (3.8) shows that both forces are proportional to the area \(F\) of the particle, but through the quantity \(N\) they also depend on its shape. As is already well known from Epstein's\textsuperscript{2} and Waldmann's\textsuperscript{4,5} work on spheres, the frictional force is proportional to the gas pressure; the thermal force is pressure independent.

In a polyatomic gas the heat flux is influenced by a magnetic field. The corresponding change in the thermal force, known as thermomagnetic force, has
been measured for disks by Adair and coworkers. The first calculation of this force has been performed by Hess for small (a ≪ l) and for large spheres (a ≫ l). A subsequent paper will be devoted to the discussion of the thermomagnetic force for spheroids, given by Eqs. (3.7), (3.8).

In a monatomic gas the forces on a particle can easily be calculated for the more general case where the coefficient α of diffuse reflection is different from 1, 0 < α ≤ 1, using.

\[ f^+ = (1 - α)f' + αf^0(n^+, T^+) \]

instead of (1.3), with the law of specular reflection

\[ f'(e) = f'(e'), \quad e' = e - 2nn⋅e \]

and

\[ f^0(n^+, T^+) = n^+(m/2πkT^+)^{3/2} \exp\{-m(1 + α/2)/2kT^+\}. \]

Then,

\[ K = -p_0\int \text{d}σ [(1 - α)nn⋅X + α/2(X + nnX)] \]

replaces Eq. (1.9), and especially for spheroidal particles we get

\[ k^+ = p_0\frac{2}{π} [4(1 + α/8)(1 - 2N) - 2α(1 - 3N)], \]

\[ k^- = p_0\frac{2}{π} [4(1 + α/8)N + α(1 - 3N)]. \]

(3.12)

and

\[ f^+ = F(4/5π) [2(1 - 2N) + α(1 - 3N)], \]

\[ f^- = F(4/5π) [2N - α/2(1 - 3N)]. \]

(3.13)

For a sphere of radius a, i.e. F = 4πa^2, N = 1/3, Eq. (3.12) reduces to Epstein’s result

\[ k^+ = k^- = \frac{2}{3}Γπp_0a, \]

(4.1)

and, according to Epstein’s formula, the thermal force depends strongly on the ratio λ_α/λ of the thermal conductivities of the particle λ_α and of the gas λ. The thermal slip number σ (occurring in the boundary condition, see Appendix) is approximately equal to 1/5, cf. Reference 12. Since the thermal force is proportional to the viscosity mean free path \( l = \frac{3}{5} \eta T p_0^{-1} \) it vanishes for high pressures with 1/p_0. The Stokes force in Eq. (4.1) is independent of pressure.

The hydrodynamic limits of Eqs. (4.1), (4.2) are valid for large particles, i.e. for a ≫ l, whereas Eqs. (3.14), (3.15) pertain to small particles with a ≪ l. For a heat conducting gas between parallel plates of distance D, Eq. (3.15) applies only if the inequality a ≪ l ≪ D is fulfilled. The free molecular flow limit is reached for a ≪ D ≪ l, with a thermal force which is proportional to pressure,

\[ f_k^+ = f_k^- = \frac{2}{3}Γπp_0a^2, \quad Q = 1/\bar{π} q^\text{trans}, \]

(4.3)

since this is true for the heat flux

\[ q^\text{trans} = \frac{1}{2}g p_0 \bar{c} T_2 - T_1 T_0 \frac{α_1^+ α_2^f}{x_1^+ + a_2^+ - α_1^+ x_2^f}. \]

(4.4)

Here, g is the unit vector directed towards the cold plate (at the temperature \( T_1 \)), and \( x_1^+ \), \( x_2^f \) are the
accommodation coefficients for translational energy at both plates. The expression (4.3) for the thermal force has been derived by Phillips for a monatomic gas. Brock’s formula contains a numerical error. Phillips also treats the intermediate regime between the Waldmann limit, Eq. (3.15), and the Knudsen limit, Equation (4.3). The decrease of the thermal force from the Waldmann limit is due to the finite plate spacing $D$. At least for small values of $l/D$ this decrease can be obtained from Eq. (3.15) by taking into account the reduction of the temperature gradient due to the temperature jump. Here ($\nabla T)_\theta$ is the temperature gradient for $\theta \rightarrow 0$, the temperature jump coefficient is of the order of 2 to 3.

5. Forces on Spheroidal Particles

In the case of a sphere the thermal force has the same direction as the negative temperature gradient, the frictional force, respectively, is parallel to the flow velocity. In contradistinction, for a spheroidal particle (e.g. a disk or a rod) the thermal force has components parallel to $-\nabla T$ and parallel to the symmetry axis $s$, i.e. in general the force has components parallel as well as perpendicular to $-\nabla T$. The unit vector parallel to $-\nabla T$ is denoted by $g$, then $Q = Q \parallel g$. Both, $s$ and $-s$ are equivalent for the characterization of the symmetry axis of the spheroid. Now, by $s$ we specifically denote that vector which encloses an acute angle $\chi$ with $q$, i.e. $\sigma \leq \chi \leq \pi/2$. Then, the unit vector $e$ perpendicular to $g$ is defined (see Figure 1a):

$$e = \frac{1}{\sin \chi} (s - \cos \chi g), \quad e \cdot g = 0,$$

$$s \cdot g = \cos \chi, \quad s \cdot e = \sin \chi.$$  

The thermal force from Eqs. (3.7), (3.8) can be rewritten as

$$K^\text{th} = K_e + K_g,$$  

$$K_e = e |Q| \left( f^\parallel - f^\perp \right) \sin \chi \cos \chi,$$

$$K_g = g |Q| \left[ f^\parallel - (f^\parallel - f^\perp) \sin^2 \chi \right].$$  

For $\chi = 0$ and $\chi = \pi/2$ the perpendicular component $K_e$ vanishes, the parallel component takes on the values $K_g = g |Q| f^\parallel$, and $K_g = g |Q| f^\perp$, respectively. In particular, if $f^\parallel$ is larger than $f^\perp$ the component $K_g$ decreases with increasing angle $\chi$, and $K_e$ has the direction $e$, reaching its maximum value $\frac{1}{2} |Q| (f^\parallel - f^\perp)$ for $\chi = \pi/4$. For $f^\parallel < f^\perp$ the component $K_e$ has the direction $-e$. In other words: Let $s$ and $g$ enclose an acute angle, then $K_e$ and $s$ enclose an acute (an obtuse) angle, if $f^\parallel$ is larger (smaller) than $f^\perp$.

The same considerations apply to the frictional force, with $g$ parallel to the flow velocity $v$.

(a) Disks

A disk of radius $a_\perp$ and thickness $2a_\parallel$ in the center is the limit of an oblate spheroid for $a_\parallel/a_\perp \rightarrow 0$. With $F = 2 \pi a_\parallel^2$ and $N = 0$ from (3.10), Eqs. (3.6), (3.8) reduce to

$$k^\parallel = 8 a_\perp^2 (1 + \pi/4) p_\theta, \quad k^\perp = 4 a_\perp^2 p_\theta = 0.280 - k^\parallel,$$

$$f^\parallel = \frac{\pi}{8} a_\perp^2, \quad f^\perp = \frac{3}{2} a_\perp^2 = 0.5 f^\parallel.$$  

In the Knudsen limit the thermal force is determined by $q^\parallel_\text{trans}$ from Eq. (4.4) and by

$$f^\parallel = \frac{1}{2} a_\perp^2 \tau, \quad f^\perp = a_\perp^2 = 0.636 f^\parallel,$$  

the expression for $f^\parallel_\perp$ has already been given by Knudsen. In all the cases of Eqs. (5.4) – (5.6) the component $K_g$ has its maximum value for $\chi = 0$, i.e. if the temperature gradient (or the flow velocity) is parallel to the normal of the disk. Correspondingly, the component $K_e$ has the same direction as $e$ (see Figure 1b).
The situation is more complicated in the hydrodynamic limit (see Appendix):
\[ k_{\parallel} = 20 a_\perp l p_0, \quad k_{\perp} = \frac{4}{3} a_\perp l p_0, \quad (5.7) \]
\[ f_{\parallel} = 10 \pi a_\parallel l \left( \frac{a_\parallel}{a_\perp} \right)^2, \quad (5.8) \]
\[ f_{\perp} = \frac{4}{3} a_\perp l \left( \frac{1 + \frac{\sigma}{4}}{a_\perp} \right), \quad (5.1) \]

For the frictional force, \( k_{h\parallel} \parallel k_{h\perp} = \frac{3}{2} \) applies, for the thermal force, however, this ratio depends strongly on the relative heat conductivity \( \Lambda = \alpha p / k \). In the limit \( \Lambda < a_\parallel / a_\perp \rightarrow 0 \) (which is probably rather unrealistic) one has again the ratio 3/2:
\[ f_{\parallel} = 20 a_\perp l \sigma = \frac{3}{2} f_{h\perp}. \quad (5.9) \]

For all realistic cases \( \Lambda > a_\parallel / a_\perp \pi / 2 \) applies, then we have
\[ f_{\parallel} = 10 \pi a_\parallel l \left( \frac{1}{a_\perp} \right), \]
\[ f_{\perp} = \frac{4}{3} a_\perp l \left( \frac{1 + \sigma}{a_\perp} \right), \quad (5.10) \]
i.e. \( f_{\parallel} \) is inversely proportional to \( \Lambda \), and (due to \( a_\parallel / a_\perp \rightarrow 0 \)) in any case much smaller than \( f_{h\perp} \). Consequently \( K_c \) has now the direction \(-e\), and already for very small angles \( \chi \) the component \( K_g \) is essentially proportional to \( \sin^2 \chi \):
\[ K_g \approx g \cdot Q \cdot f_{h\perp} \sin^2 \chi. \]

This strange behavior is due to the fact that, in the hydrodynamic limit, the thermal force is proportional to the temperature difference over the disk, hence (for \( \Lambda + 0 \)) proportional to \( a_\parallel \) and \( \Lambda^{-1} \) for \( f_{h\parallel} \), and proportional to \( a_\perp \) (and very weakly dependent on \( \Lambda \)) for \( f_{h\perp} \), respectively. It is rather doubtful, however, whether or not these limits can be met experimentally.

(b) Rods

For a very long prolate spheroid \( (a_\perp / a_\parallel \rightarrow 0) \) Eq. (3.11) gives \( F = \pi a^2 a_\parallel a_\perp, N = 3 \). The forces on such a rod (length \( 2 a_\parallel \), radius \( a_\perp \)) are obtained from Eqs. (3.6), (3.8):
\[ k_{\parallel} = 2 \pi a_\parallel a_\perp p_0, \quad k_{\perp} = 3 \pi a_\parallel a_\perp (1 + \pi/6) p_0 = 2.285 k_{\parallel}, \quad (5.11) \]
\[ f_{\parallel} = \frac{3}{8} \pi a_\parallel a_\perp, \quad f_{\perp} = \frac{3}{8} \pi a_\parallel a_\perp = \frac{3}{8} f_{\parallel}. \quad (5.12) \]

Here, for both forces the component \( K_g \) has its maximum value for \( \chi = \pi / 2 \), i.e. if the axis of the rod is perpendicular to \( g \). As a consequence, \( K_e \) has the same direction as \(-e\) (see Figure 1 e). The same qualitative behavior is found for the frictional force in the hydrodynamic limit:
\[ k_{h\parallel} = \frac{5 \pi a_\parallel l}{\ln(a_\parallel / a_\perp)} p_0, \quad k_{h\perp} = 2 k_{h\parallel}. \quad (5.13) \]

For the thermal force, however, the situation is more complicated:
\[ f_{h\parallel} = \frac{5 \pi a_\parallel l}{\ln(a_\parallel / a_\perp)} \sigma, \quad f_{h\perp} = \frac{2}{\Lambda + 1} f_{h\parallel}. \quad (5.14) \]

In particular, \( f_{h\parallel} \) is independent of \( \Lambda \), whereas \( f_{h\perp} \) is proportional to \((\Lambda + 1)^{-1}\), for moderate values of \( \Lambda \) both are of the same order of magnitude. Notice, \( f_{h\parallel} \approx 2 f_{h\parallel} \) for \( \Lambda \ll 1 \), \( f_{h\perp} = f_{h\parallel} \) for \( \Lambda = 1 \), and \( f_{h\parallel} < f_{h\parallel} \) for \( \Lambda > 1 \).

Hence, the component \( K_e \) is parallel to \(-e\) for \( \Lambda < 1 \), and parallel to \( e \) for \( \Lambda > 1 \), for \( \Lambda = 1 \) it is not present at all.

In this paper, the "Waldmann limits" (i.e. \( a \ll l < D \)) of the frictional and thermal forces have been calculated for spheroidal particles suspended in a polyatomic gas. The results have been compared to the corresponding expressions in the hydrodynamic limit \((l \ll a \ll D)\), and, for the thermal force, also to the formulae available in the free molecular flow limit. In the slip flow regime \((l \ll a \ll \lambda)\) a lot of work has been done for spheres, see e.g. References 12–15. Calculations for the thermal force on a sphere in the transition regime from the Waldmann limit to the Knudsen limit are due to Phillips. For nonspherical particles, however, neither of these regions has been treated up to now.

Appendix

The Forces in the Hydrodynamic Limit

For convenience, some of the well known hydrodynamic expressions are stated. The force on the ellipsoid
\[ \sum_{i=1}^{3} x_i^2/\alpha_i^2 = 1 \]
is written as
\[ K = 6 \pi \eta \sum_{i=1}^{3} R_i e_i e_i V, \quad (A1) \]
where \( \mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3 \) form an orthonormal basis of three-dimensional space. For the frictional force \( K^\gamma \), the vector \( \mathbf{V} \) is equal to the gas velocity \( \mathbf{v}_\infty \) far away from the particle, \( \mathbf{V} = \mathbf{v}_\infty \). And, according to Lamb \(^1\), the length \( R_i \) has to be replaced by

\[
R_i^\gamma = \frac{\delta a_1 a_2 a_3}{z + \alpha_i a_i^2},
\]

with

\[
z = a_1 a_2 a_3 \int_0^\infty d\lambda \left[ (a_2^2 + \lambda) (a_2^2 + \lambda) (a_3^2 + \lambda) \right]^{-1/2},
\]

\[
a_i = a_1 a_2 a_3 \int_0^\infty d\lambda \left[ (a_2^2 + \lambda) \right]^{-1/2} \left( a_2^2 + \lambda \right) (a_3^2 + \lambda) \right]^{-1/2}
\]

Lamb's method \(^1\) can also be applied for the calculation of the thermal force \( K^\text{th} \), using the boundary condition of thermal slip in the form

\[
\mathbf{v}^{\text{tan}} = \frac{z^{\text{trans}}}{\rho_0} \left( \nabla T \right)^\infty.
\]

Here \( z^{\text{trans}} \) is the translational heat conductivity of the gas, and \( \sigma \) is the thermal slip number. Typically, \( \sigma \) is approximately equal to 1/5, the number first given by Maxwell (see also Ref. 12). Then, in Eq. (A1) the vector \( \mathbf{V} \) has to be replaced by

\[
\mathbf{V} = -\sigma \left( z^{\text{trans}}/\rho_0 \right) \left( \nabla T \right)^\infty,
\]

and the length \( R_i \) by

\[
R_i^\text{th} = R_i^\gamma \frac{2 - a_i}{2 - a_i (1 - A)}.
\]

By \( \left( \nabla T \right)^\infty \) the temperature gradient in the gas, far away from the ellipsoid is denoted, \( A = \lambda_p/\lambda \) is the ratio of the heat conductivities of the particle \( \lambda_p \) and of the gas \( \lambda \). Notice, \( R_i^\text{th} \rightarrow R_i^\gamma \) for either \( \alpha_i \rightarrow 0 \) or \( A \rightarrow 0 \).

For an oblate spheroid the thermal force has been calculated by Sexl \(^1\); in particular, he treated the case that \( \left( \nabla T \right)^\infty \) is parallel to \( \mathbf{e}_1 \), and expressed \( K^\text{th} \) by \( R_i^\gamma \) by and the temperature difference

\[
\delta T = \frac{4 a_1}{2 - a_i (1 - A)} \mathbf{e}_1 \cdot \left( \nabla T \right)^\infty
\]

between the poles \( x_1 = a_1, x_2 = x_3 = 0 \) and \( x_1 = -a_1, x_2 = x_3 = 0 \) of the ellipsoid. A different approach for the thermal force on a flat disk is due to Epstein \(^7\).

If the ellipsoid has rotational symmetry about the axis \( \mathbf{s} = \mathbf{e}_1 \), with \( a_\parallel \approx a_1 \), \( a_\perp \approx a_2 = a_3 \), and \( R_\parallel \approx R_1, R_\perp \approx R_2 = R_3 \). Equation (A1) can be rewritten in the form

\[
K = 6 \pi \eta \left[ R_\parallel \mathbf{ss} \cdot \mathbf{V} + R_\perp \left( \mathbf{V} - \mathbf{ss} \cdot \mathbf{V} \right) \right].
\]

Analytic expressions for \( \chi \) and \( \alpha_\parallel \equiv \alpha_1, \alpha_\perp \equiv \alpha_2 = \alpha_3 \) can be found in a paper by Gans \(^19\). The result for oblate spheroids is

\[
\chi = 2 a_2^2 (1/\varepsilon) \arcsin \varepsilon,
\]

\[
a_\parallel = \left( 2/\varepsilon^2 \right) \left[ 1 - (1/\varepsilon) \arcsin \varepsilon \right],
\]

\[
a_\perp = \left( 1/\varepsilon^2 \right) \left[ (1/\varepsilon) \arcsin \varepsilon - (1 - \varepsilon^2) \right],
\]

with \( \varepsilon = 1 - (a_\perp^2/a_\parallel^2)^{1/2} \). For prolate spheroids we have \( \varepsilon = (1 - a_\perp^2/a_\parallel^2)^{1/2} \) and

\[
\chi = a_\parallel^2 - \frac{1 - \varepsilon}{2 \varepsilon^3} \ln \frac{1 + \varepsilon}{1 - \varepsilon},
\]

\[
a_\parallel = \frac{1 - \varepsilon}{\varepsilon^2} \left[ \frac{1}{\varepsilon} \ln \frac{1 + \varepsilon}{1 - \varepsilon} - 2 \right],
\]

\[
a_\perp = \frac{1 - \varepsilon}{\varepsilon^2} \left[ \frac{1}{\varepsilon} \ln \frac{1 + \varepsilon}{1 - \varepsilon} - 2 \right].
\]

In the case of a sphere of radius \( a \), Eqs. (A5), (A6) reduce to

\[
\chi = 2 a_\parallel^2, \quad \alpha_\parallel = \alpha_\perp = 2/3.
\]

In particular, for a disk \( (a_\parallel/a_\perp \rightarrow 0) \) Eq. (A5) yields

\[
\chi = \pi a_\parallel a_\perp, \quad \alpha_\parallel = 2 - \pi (a_\parallel/a_\perp), \quad \alpha_\perp = \frac{1}{2} \pi a_\parallel/a_\perp.
\]

and Eq. (A6) gives for a rod \( (a_\perp/a_\parallel \rightarrow 0) \)

\[
\chi = 2 a_\parallel^2 \ln (a_\parallel/a_\perp), \quad \alpha_\parallel = 2 \left( a_\parallel^2/a_\perp^2 \right) \ln (a_\parallel/a_\perp), \quad \alpha_\perp = 1 - \left( a_\perp^2/a_\parallel^2 \right) \ln (a_\parallel/a_\perp).
\]

Acknowledgements

This research was supported by the Robert A. Welch Foundation, Houston, Texas. The author would like to express appreciation to Professor T. W. Adair, III for stimulating discussions and useful comments. Thanks are also due to Dr. L. A. Davis for his valuable assistance.

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