Expressions which are exact in the distorted wave Born approximation are derived for a number of kinetic theory collision cross sections, viz., $\sigma^{(\text{so})}$, $\sigma^{(\text{oo})}$, $\sigma^{(\text{qq})}$ and $\sigma^{(\text{qq})}$. This is done for a number of nonspherical potentials, namely for single-$P_1$, single-$P_2$, dipole-dipole, dipole-quadrupole and quadrupole-quadrupole interactions. A number of additional approximations is used to obtain approximate relations between some of these cross sections and to derive an approximate law of corresponding states. A simple model is described which makes the approximate evaluation of collision cross sections for multipole-multipole interactions very straightforward.

1. Introduction

The number of transport and relaxation phenomena in dilute polyatomic gases that has been theoretically elucidated by the use of the Waldmann-Snider equation has steadily increased in the past decade. The results of all such treatments are expressed in terms of collision integrals, i.e., of matrix elements of the Waldmann-Snider collision super-operator. Although some work already has been done on the connection between such collision integrals and the nonspherical intermolecular potential, many questions remain unanswered. It is the purpose of the present article to derive a number of expressions for certain collision integrals which are as exact as possible (for linear molecules). From these formulae a number of more approximate expressions are then derived by means of various (approximate) simplifications. For a list of the various subjects treated in this article, see the Summary (Section 6).

The collision integrals treated in this article will now be discussed in their relation to experiment.

a) The collision cross section $\sigma^{(\text{oo})}$ determines the bulk viscosity $\eta_B$ of the gas through the equation

$$\eta_B = k T [\sigma^{(\text{oo})} \langle \mathbf{v}_{\text{rel}} \cdot \mathbf{v}_{\text{rel}} \rangle \langle \mathbf{v}_{\text{rel}} \cdot \mathbf{v}_{\text{rel}} \rangle - \frac{9}{2} \left( \frac{k T}{\epsilon_{\text{int}}} \right)^2]^{-1},$$

(1.1)

where $\langle \mathbf{v}_{\text{rel}} \rangle_0$ is an average relative velocity equal to $(16 k T/\pi m)^{1/2}$ and $\epsilon_{\text{int}}$ is the internal heat capacity per molecule.

b) The collision cross section $\sigma^{(\text{qq})}$ determines the $\eta_3$ value for which the Senftleben-Beenakker effect reaches half of its saturation value. Here $\eta_3$ is one of the shear viscosity coefficients of the De Groot-Mazur scheme and $\eta_0$ is the field-free shear viscosity coefficient. The relation mentioned above is given as

$$\eta_3 = \frac{H}{p} \langle \mathbf{v}_{\text{rel}} \cdot \mathbf{v}_{\text{rel}} \rangle \sigma^{(\text{qq})} \left[ \frac{g_{\text{rot}}}{2} \mu_N k T \right],$$

(1.2)

where $g_{\text{rot}}$ is the rotational $g$-factor and $\mu_N$ the nuclear magneton.

c) The collision cross section $\sigma^{(\text{qq})}$ determines the saturation value of $\eta_3/\eta_0$ via the relation

$$\eta_3/\eta_0 = \sigma^{(\text{qq})} / \sigma^{(\text{qq})},$$

(1.3)

with $\sigma^{(\text{qq})}$ and $\sigma^{(\text{qq})}$ related to $\eta_0$ by

$$\eta_0 = k T [\langle \mathbf{v}_{\text{rel}} \cdot \mathbf{v}_{\text{rel}} \rangle \langle \mathbf{v}_{\text{rel}} \cdot \mathbf{v}_{\text{rel}} \rangle - \frac{9}{2} \left( \frac{k T}{\epsilon_{\text{int}}} \right)^2]^{-1},$$

(1.4)

The collision integrals $\sigma^{(\text{qq})}$ and $\sigma^{(\text{qq})}$ occur in the theory of nuclear magnetic resonance in diatomic gases. See Refs. 8 and 9.

e) The cross section $\sigma^{(\text{qq})}$ does not enter into the theory of any easily measurable phenomenon (for one example, see Ref. 10). It has been included here for completeness.
2. Derivation of General Expressions for Diagonal Collision Integrals

The general expression for a collision integral as derived from the Waldmann-Snider equation reduces somewhat if the collision integral is diagonal. It is in that case given by

\[ (A, \mathcal{D} A) = (2\pi)^4 h^2 \text{tr}_1\text{tr}_2 \int f_1(0) f_2(0) A_1^\dagger t g g^\dagger \{ (A_1 + A_2) t g g^\dagger \} \{ (A_1 + A_2) t g g^\dagger \} \delta(E) \delta(P) \, dp_1 \, dp_2 \, dp'_1 \, dp'_2. \]  

Here, \( t g^\dagger \) and \( t g^\dagger \) are matrix elements of the transition operator \( t \) and its adjoint, respectively, in (relative) momentum space, \( g \) and \( g' \) denoting the relative velocities before and after a collision. These matrix elements are still operators in internal state space. A prime indicates a dependence on \( p_1' \) and \( p_2' \), the postcollisional, rather than on \( p_1 \) and \( p_2 \), the precollisional linear momenta of the colliding molecules. The Dirac delta functions express the conservation of total linear momentum and total energy, while the symbols \( \text{tr}_1 \) and \( \text{tr}_2 \) indicate that the traces over internal states of the two molecules are to be performed.

The superscript „\( \dagger \)“ denotes a tensorial transpose, while the dagger (\( \dagger \)) indicates the Hermitean conjugate. The equilibrium distribution function, \( \langle 0 \rangle \), is, for linear molecules, given by

\[ \langle 0 \rangle = \frac{Q}{\sqrt{2\pi}} \exp\left\{-\beta j(j+1)\right\} \frac{1}{A} \exp\left(-\frac{W^2}{2}\right), \]  

where \( g(j) \) is the weight of level \( j \) due to nuclear spin states (e.g., \( g(2k) = 1 \) and \( g(2k+1) = 3 \) for normal hydrogen), \( B = \frac{h^2}{2I k T} \) with \( I \) the moment of inertia of the molecule, \( Q = \sum g(j) (2j+1) \exp\{-B j(j+1)\} \) is the rotational partition function and \( W \) is the reduced peculiar velocity, \( W = \frac{p_j}{(2m k T)^{1/2}} \).

The form of the transition operator \( t \) in the first order distorted wave Born approximation (DWBA) has already been discussed in the literature\(^{4,11,12,13-15}\).

The result may be summarized as follows: for a potential of the form (see refs.\(^{13-15}\) for the Cartesian tensor notation)

\[ V(r_1, r_2, R) = \sum_{l_1, l_2, L} V_{l_1 l_2 L}(R) [r_1^{(l_1)} [r_2^{(l_2)} \phi_{l_1 + l_2} T(l_1, l_2, L) \otimes L(R)], \]  

the DWBA transition operator has the form

\[ t g = \sum_{l_1, l_2, L} t g (l_1, l_2, L), \]  

\[ t g (l_1, l_2, L) = [r_1^{(l_1)} [r_2^{(l_2)} \phi_{l_1 + l_2} T(l_1, l_2, L) \otimes L A^{(L)}(l_1, l_2, g, g'). \]  

In these equations \( r_i \) is the unit vector along the internuclear axis of the \( i \)-th molecule, while \( R \) (absolute value \( R = R/R \)) is the vector extending from the center-of-mass of molecule 1 to the center-of-mass of molecule 2. Further, \( T(l_1, l_2, L) \) is a Cartesian (3 - \( j \))-tensor\(^{15}\), while the symbol \( \otimes k \) indicates a \( k \)-fold contraction. The \( L \)-th rank tensor \( A^{(L)}(l_1, l_2, g, g') \) is given by [for \( (l_1, l_2, L) \neq (0, 0, 0) \)]:

\[ A^{(L)}(l_1, l_2, g, g') = \int dR \chi^{(+)}_k(R) V_{l_1 l_2 L}(R) [R^+]^L \chi^{(+)}_k(R). \]  

Here, \( \chi^{(+)}_k(R) \) is the solution of the Schrödinger equation for the spherical part \( V_0(R) \equiv V_{000}(R) \) of the intermolecular potential determined by the plane wave \( \hbar^{-3/2} \exp\{ik \cdot R\} \) and the outgoing scattering condition. The wavevector \( k \) is related to the relative velocity \( g \) by

\[ k = \frac{1}{2} m g/h. \]  

It is convenient to define „matrix elements“ of collision integrals between sets of „potential quantum numbers“ \( (l_1, l_2, L) \) by the equation

\[ \langle l_1 l_2 L | (A, \mathcal{D} A) | l_1' l_2' L' \rangle = (2\pi)^4 h^2 \text{tr}_1\text{tr}_2 \int f_1(0) f_2(0) A_1^\dagger \times \{ (A_1 + A_2) t g (l_1, l_2, L) t g (l_1', l_2', L') - t g (l_1, l_2, L) (A_1 + A_2) t g (l_1', l_2', L') \} \delta(E) \delta(P) \times dp_1 dp_2 dp_1' dp_2'. \]  

(2.8)
In what follows only ,,diagonal in potential,, matrix elements, i.e., matrix elements with \((l_1, l_2, L) = (l_1', l_2', L')\), will be considered, the reason being that for the potentials of interest here no cross terms occur (see Section 3 for a discussion on this point).

It is now possible to consider the various collision integrals of interest in this article in detail.

a) The collision integral \([0010]_{0010}\).

This collision integral is equal\(^1\) to \([0001]_{0000}\) for which \(A = J^2 = j(j + 1)\). Inserting this and Eq. (2.5) into Eq. (2.8) yields after some calculation

\[
\langle l_1 l_2 L | [0010]_{0010} | l_1 l_2 L \rangle = \Omega(l_1, l_2, L) B Q^2 \sum_{j_1} g(j_1) g(j_2) B^{(l_1)}_{j_1 j_1} B^{(l_2)}_{j_2 j_2} (j_1 + 1) e \exp(-E) \sigma_{11} l_2 L(\epsilon) \langle \nu_{\text{rel}} \rangle
\]

(2.9)

Here \(\Omega(l_1, l_2, L)\) is the contracted square of \(T(l_1, l_2, L)\) (see Appendix A), while \(B^{(l)}_{j j'}\) is defined by (trdeg indicates a trace over degenerate internal states)

\[
\text{trdeg} \{ \rho_j \} \rho_j \rho_j' = B^{(l)}_{j j'} T(l, 0, l).
\]

Further, \(\epsilon\) and \(E\) are given by

\[
\epsilon = B[j_1(j_1 + 1) + j_2(j_2 + 1) - j_1'(j_1' + 1) - j_2'(j_2' + 1)]
\]

and

\[
E = B[j_1(j_1 + 1) + j_2(j_2 + 1)],
\]

while the cross section \(\sigma_{11} l_2 L(\epsilon)\) has been defined by

\[
\sigma_{11} l_2 L(\epsilon) = \langle \nu_{\text{rel}} \rangle 0 (2 L + 1) \int \int h_1(0,0) h_2(0,0) A^{(l_1, l_2)}(g, g') \otimes L A^{(l_1, l_2)}(g, g') \delta(E) \delta(P) \, dp_1 \, dp_2 \, dp_1' \, dp_2'.
\]

(2.13)

The projection operators \(\rho_j\) occurring in Eq. (2.10) project onto the \((2j + 1)\)-dimensional manifold of degenerate states belonging to the quantum number \(j\) and have been introduced by the identity

\[
[r]^{-1} = \prod_{L, j'} \rho_j r \rho_j'.
\]

The quantity \(B^{(l)}_{j j'}\) is given in Appendix A together with the derivation of Eq. (2.10).

b) The collision integrals \([1000]_{0101}\) and \([0100]_{1001}\).

These collision integrals are defined by

\[
\text{trdeg} \{ \rho_j \} \rho_j = \frac{1}{2} (J \cdot R(\nu) J).
\]

(2.15)

The collision integral containing \(\rho_j\) is obtained from Eq. (2.1) by dropping the terms \(A_2\) and \(A_2'\) from it. Using the methods sketched in Appendix A, a general result for the collision integrals \([0400]^{(c)}\), defined by

\[
[0400]^{(c)} = ([J]^{(k)} \otimes k, R(\nu)[J]^{(k)})/(2k + 1)
\]

may be derived. This is given as

\[
\langle l_1 l_2 L | [0400]^{(c)} | l_1 l_2 L \rangle = \Omega(l_1, l_2, L) [(2 k + 1) Q^2]^{-1} \sum_{j_1} g(j_1) g(j_2) B^{(l_1)}_{j_1 j_1} B^{(l_2)}_{j_2 j_2} (j_1 + 1) e \exp(-E) \sigma_{11} l_2 L(\epsilon) \langle \nu_{\text{rel}} \rangle
\]

(2.17)

and

\[
\langle l_1 l_2 L | [0400]^{(c)} - [0400]^{(c)} | l_1 l_2 L \rangle = \Omega(l_1, l_2, L) [(2 k + 1) Q^2]^{-1} \sum_{j_1} g(j_1) g(j_2) B^{(l_1)}_{j_1 j_1} B^{(l_2)}_{j_2 j_2} (j_1 + 1) e \exp(-E) \sigma_{11} l_2 L(\epsilon) \langle \nu_{\text{rel}} \rangle
\]

(2.18)
In these equations, the \( \left[ \begin{array}{ccc} a & b & c \\ d & e & f \end{array} \right] \) are \((6-j)\)-symbols, while the \( \zeta_j^{(k)} \) are discussed in Appendix A.

For the special value \( k=1 \) a further calculation, using the explicit form of the \( \zeta_j^{(k)} \), the \( \Omega(j,k,l) \) and the \((6-j)\)-symbols gives the results

\[
\langle l_1 l_2 L \mid \left[ \begin{array}{ccc} 0 & 1 & 0 \\ 0 & 0 & 0 \end{array} \right] \mid l_1 l_2 L \rangle = \frac{1}{12} \Omega(l_1, l_2, L) Q^{-2} \sum_{j,j'} \langle j | j' \rangle B_{j,j'}^{(l_1)} B_{j,j'}^{(l_2)} \exp(-E) \sigma_{12L}(e) \langle \psi_{rel} \rangle_0 .
\]

From Eqs. (2.19) and (2.20) the interesting relation (see Section 3 for special cases)

\[
\langle l_1 l_2 L \mid \left[ \begin{array}{ccc} 0 & 1 & 0 \\ 0 & 0 & 0 \end{array} \right] \mid l_1 l_2 L \rangle + \langle l_2 l_1 L \mid \left[ \begin{array}{ccc} 0 & 1 & 0 \\ 0 & 0 & 0 \end{array} \right] \mid l_2 l_1 L \rangle = [l_1(l_1 + 1) + l_2(l_2 + 1)] [L(L + 1)]^{-1}
\times \left\{ \langle l_1 l_2 L \mid \left[ \begin{array}{ccc} 0 & 1 & 0 \\ 0 & 0 & 0 \end{array} \right] \mid l_1 l_2 L \rangle + \langle l_2 l_1 L \mid \left[ \begin{array}{ccc} 0 & 1 & 0 \\ 0 & 0 & 0 \end{array} \right] \mid l_2 l_1 L \rangle \right\}
\]

is easily derived.

c) The collision integrals \( \langle 0 | 0 | 0 \rangle \) and \( \langle 0 | 0 | 0 \rangle \).

From Eqs. (2.17) and (2.18) it follows that

\[
\langle l_1 l_2 L \mid \left[ \begin{array}{ccc} 0 & 1 & 0 \\ 0 & 0 & 0 \end{array} \right] \mid l_1 l_2 L \rangle = (1/60) \Omega(l_1, l_2, L) Q^{-2} \sum_{j,j'} \langle j | j' \rangle B_{j,j'}^{(l_1)} \exp(-E)
\times \left\{ 3l_1(l_1 + 1) [4j_j^2 + 4j_j - (l_j^2 + l_j + 1)] + A_{12}^2 + M'(l_1, l_2, L) A_{12} \right\} \sigma_{12L}(e) \langle \psi_{rel} \rangle_0 .
\]

Here the \( A_j \) and the \( M'(l_1, l_2, L) \) are given by

\[
A_j = j_j(j_j + 1) - j'_j(j'_j + 1),
\]

\[
M'(l_1, l_2, L) = \frac{1}{2} [3X(X - 1) - 4l_1(l_1 + 1)l_2(l_2 + 1)][l_1(l_1 + 1) + l_2(l_2 + 1)]^{-1}
\]

with

\[
X = l_1(l_1 + 1) + l_2(l_2 + 1) - L(L + 1)
\]

and

\[
M'(l_1, l_2, L) = 0 .
\]

Equations (2.9), (2.19—21) and 2.23) are the basic formulæ from which a calculation of these collision integrals has to be started. It will now be necessary to have a closer look at the cross section \( \sigma_{12L}(e) \), in which the collision integrals have been expressed. The starting point for this is eq. (2.6). The scattering wave functions \( \chi^{(+)}(R) \) occurring there may be expanded in partial waves in the standard way; in Cartesian tensor notation this expansion takes the form

\[
\chi^{(+)}(R) = h^{3/2} \sum_{m=0}^{\infty} \chi_m(k | R) [k^\cdot]^m \otimes [R^\cdot]^m
\]

with

\[
\chi_m = [(2m + 1)!k^2m(m!)^2] i^m \exp(i \eta_m) ,
\]

where \( y_m \) is the phase shift. \( \chi_m \) has been chosen so that \( y_m \) has the asymptotic form

\[
y_m(k | R) \approx \sin(k R - \frac{1}{2} m x + \eta_m) .
\]

Insertion of this expansion in Eq. (2.6) yields (using the results of Appendix A)

\[
\mathcal{A}_{12L}(g, g') = h^3 \sum \zeta_m \chi_m(-1)^m \beta_{12L}(\gamma, \gamma') \times \mu(m_1, L, m_2)
\times [\mathbf{Y}^\cdot]^m \otimes [\mathbf{T}(m_1, L, m_2) \otimes [\mathbf{Y}^\cdot]^m] ,
\]

where instead of \( k \) or \( g \) the dimensionsless vector \( \mathbf{Y} \) has been introduced by the relation

\[
\gamma = h(m k T)^{-1/2} \mathbf{r} .
\]

The quantities \( \beta_{12L}^{(m)}(\gamma, \gamma') \) are given by

\[
\beta_{12L}^{(m)}(\gamma, \gamma') = \int \mathcal{R} y_m(k | R) V_{12L}(R) y_m(k' | R) .
\]

Equation (2.30) may be inserted into Eq. (2.13). After changing the integration variables from the \( \mathbf{p}'s \) to \( \mathbf{Y} \) and the dimensionless center-of-mass velocity \( \mathbf{G} \) by means of the transformation (see the definition following Eq. (2.2) for \( W \))

\[
\mathbf{W}_1 = 2^{-1/2}(G - \mathbf{Y}) ; \quad \mathbf{W}_2 = 2^{-1/2}(G + \mathbf{Y}) ; \quad \mathbf{W}_1' = 2^{-1/2}(G - \mathbf{Y}) ; \quad \mathbf{W}_2' = 2^{-1/2}(G + \mathbf{Y}) ,
\]

a number of integrations may be performed with the final result

\[
\sigma_{12L}(e) = \int d\gamma \gamma^2 \gamma' H_{12L}(\gamma, \gamma') e^{-\gamma} ,
\]

where \( \mathbf{W} \) is a number of collision integrals.
\[ H_{112L}(\gamma, \gamma') = m^2 [32 \pi^3 h^4 (2L + 1)]^{-1} \]
\[ \times \sum_{m_1, m_2} |m_1|^2 |m_2|^2 \mu_2(m_1, L, m_2) \times \] (2.35)
\[ \times \mu(m_1, 0, m_2) \mu(m_2, 0, m_2) \Omega(m_1, L, m_2) \times [\hat{\beta}_{112L}(\gamma, \gamma')]^2. \]

It is now in principle possible to completely calculate a collision integral, the necessary steps being:
a) Solve the partial wave equations for \( V_0(R) \).
b) Use these solutions to calculate the \( \hat{\beta}_{112L}(\gamma, \gamma') \) and \( H_{112L}(\gamma, \gamma') \).
c) Calculate \( \sigma_{112L}(\epsilon) \) and from this the collision integrals using Eqs. (2.9), (2.19–21) and (2.23).

Step c) may be simplified by making use of the following symmetry property: since [see Eq. (2.32)]
\[ \hat{\beta}_{112L}(\gamma, \gamma') = \hat{\beta}_{112L}(\gamma', \gamma) \] (2.36)
it follows that [see Eq. (2.35)]
\[ H_{112L}(\gamma, \gamma') = H_{112L}(\gamma', \gamma). \] (2.37)

Further, energy conservation requires that \( \gamma'^2 = \gamma^2 + \epsilon \), so that if \( \epsilon < 0 \) the integral in Eq. (2.34) does not really have zero as its lower limit, but \( (\epsilon)^{1/2} \).

Using this and the symmetry relation (2.37) it is easily shown that the functional dependence of \( \sigma_{112L}(\epsilon) \) may be represented by
\[ \sigma_{112L}(\epsilon) = \exp(\epsilon/2) I_{112L}(|\epsilon|). \] (2.38)

This equation has been used in the next section to simplify the expressions for the collision integrals for specific potentials.

Instead of the collision integrals themselves, the collision cross sections which may be derived from them have been used in what follows. The definitions of these cross sections for the cases of interest here are \[ \sigma_{112L}^{(0010)} = \frac{3}{2} \langle V_{rel} \rangle_0^{-1} \] (0010) \[ \sigma_{112L}^{(0100)}(\gamma') = 3 \langle J^2 \rangle_0 \langle V_{rel} \rangle_0^{-1} \] (0100) \[ \sigma_{112L}^{(0200)}(\gamma') = \frac{15}{2} \langle J^2 (J^2 - \frac{3}{4}) \rangle_0 \langle V_{rel} \rangle_0^{-1} \] (0200) \[ \sigma_{112L}^{(0200)}(\gamma') \] (2.39)

### 3. Expressions for a Number of Potentials

From the general formulae for the diagonal cross sections derived in the previous section expressions for particular potentials may be extracted. To this end it is only necessary to consider the various possible values that \( j_1' \) and \( j_2' \) can take, given values of \( j_1 \) and \( j_2 \). For the potentials considered in this section, these selection rules are listed in Table I. They may all be obtained from the factor \( \mu(j_1, l_1, l_1') \mu(j_2, l_2, j_2') \) occurring in each collision integral (Appendix A). This factor is zero unless the indices satisfy certain conditions given in Appendix A.

<table>
<thead>
<tr>
<th>Potential quantum numbers</th>
<th>Potential Symbol</th>
<th>( \Delta j_1 )</th>
<th>( \Delta j_2 )</th>
</tr>
</thead>
<tbody>
<tr>
<td>( l_1 ) ( l_2 ) ( L )</td>
<td>( \sigma_{112L} )</td>
<td>( j_1 ) ( j_2 )</td>
<td>( j_1' ) ( j_2' )</td>
</tr>
<tr>
<td>0 \ 0 \ 0</td>
<td>SPH</td>
<td>0 \ 0</td>
<td>0 \ 0</td>
</tr>
<tr>
<td>1 \ 0 \ 1</td>
<td>P1</td>
<td>( \pm 1 ) \ 0</td>
<td>0 \ 0</td>
</tr>
<tr>
<td>0 \ 1 \ 1</td>
<td>DD</td>
<td>0 \ ( \pm 2 )</td>
<td>0 \ 0 \ ( \pm 2 )</td>
</tr>
<tr>
<td>2 \ 0 \ 2</td>
<td>P2</td>
<td>0 \ ( \pm 2 )</td>
<td>0 \ 0 \ ( \pm 2 )</td>
</tr>
<tr>
<td>1 \ 1 \ 2</td>
<td>DD</td>
<td>0 \ ( \pm 2 )</td>
<td>0 \ 0 \ ( \pm 2 )</td>
</tr>
<tr>
<td>2 \ 1 \ 3</td>
<td>DQ</td>
<td>( \pm 1 ) \ 0 \ ( \pm 2 )</td>
<td>0 \ 0 \ ( \pm 2 )</td>
</tr>
<tr>
<td>2 \ 2</td>
<td>QQ</td>
<td>0 \ 0 \ ( \pm 2 )</td>
<td>0 \ 0 \ ( \pm 2 )</td>
</tr>
</tbody>
</table>

However, it may be shown quite generally (in the same way as the expressions in the previous section were derived) that \( \sigma_{112L}^{(0010)} \), \( \sigma_{112L}^{(0100)} \), \( \sigma_{112L}^{(0200)} \) and \( \sigma_{112L}^{(0200)} \) are strictly diagonal in potential quantum numbers and that \( \sigma_{112L}^{(0200)} \) is diagonal in the restricted set of Table I.

In the following pages, the results for several potentials are listed. In order to make reference to these equations easier, a special numbering system is used, which includes the potential symbol (see Table I for these). Explicit expressions for the dipole-quadrupole and quadrupole-quadrupole interactions have also been obtained. These are not listed here because they are too lengthy.

Also listed are expressions in the high temperature approximation. In this approximation \( I_{112L}(\epsilon) \) is replaced by \( I_{112L}(\epsilon_0) \). The remaining sums are
then calculated in the classical limit, i.e., they are replaced by integrals. The averages needed to compute $\langle \varepsilon \rangle_0$ are:

\[
\langle j_1 \rangle_0 = \frac{1}{2} (\pi B)^{1/2},
\]
\[
\langle j_1 - j_2 \rangle_0 = \frac{1}{2} (2 - \frac{1}{2}) (\pi B)^{1/2},
\]
\[
\langle j_1 - 2 j_2 \rangle_0 = \frac{1}{2} (3 - \frac{24}{25} \frac{1}{5}) (\pi B)^{1/2}.
\] (3.1)

In all expressions obtained only the lowest power of $B$ is retained. The high temperature approximation is expected to be rather good since $\varepsilon$ is proportional to $T^{-1}$, i.e., at high temperatures the dependence of $I_{112L}(\varepsilon)$ on $\varepsilon$ is weak. Therefore replacement of $\varepsilon$ by its average value will be reasonable for many values of $j$.

A) Single-$P_1$ potential (P1)

a) Exact DWBA expressions:

\[
\sigma_{^{0010}} = (8/3) B^2 Q^{-1} \sum_{j=0}^{\infty} (j+1)^3
\]
\[
\times \exp\left[-B(j+1)^2\right] I_{101}(2 B(j+1)).
\] (P1-1)

b) High temperature expressions:

\[
\sigma_{^{0010}} = \frac{4}{3} B I_{101}(1/\sqrt{\pi} B).
\] (P1-4)

\[
\sigma_{^{0100}} = B I_{101}(1/\sqrt{\pi} B).
\] (P1-5)

\[
\sigma_{^{0200}} = \frac{2}{5} B I_{101}(1/\sqrt{\pi} B).
\] (P1-6)

B) Single-$P_2$ potential (P2)

a) Exact DWBA expressions:

\[
\sigma_{^{0010}} = (8/3) B^2 Q^{-1} \sum_{j=0}^{\infty} g(j) (j+1) (j+2) (2j+3) \exp\{-B(j^2 + 3j + 3)\} I_{202}(2 B(2j+3)).
\] (P2-1)

\[
\sigma_{^{0100}} = 2 \left[ Q \langle J^2 \rangle_0^{-1} \sum_{j=0}^{\infty} g(j) (j+1) (2j+1) (2j+3) \exp\{-B(j^2 + 3j + 3)\} I_{202}(0) \right]
\]
\[
+ 3 \sum_{j=0}^{\infty} g(j) (j+1) (j+2) 2j+3 \exp\{-B(j^2 + 3j + 3)\} I_{202}(2 B(2j+3))
\] (P2-2)

\[
\sigma_{^{0200}} = \frac{1}{2} Q \langle J^2 \rangle_0^{-1} \left[ 3 Q \langle 4j^2 + 4j - 7 \rangle_0 (j+1) (2j+1) (2j+3) \right] I_{202}(0)
\]
\[
+ 3 \sum_{j=0}^{\infty} g(j) (j+1) (j+2) 2j+3 \exp\{-B(j^2 + 3j + 3)\}(44j^2 + 132j + 62) I_{202}(2 B(2j+3))
\] (P2-3)

b) High temperature expressions:

\[
\sigma_{^{0010}} = (8/3) B I_{202}(2/\sqrt{\pi} B),
\] (P2-4)

\[
\sigma_{^{0100}} = \frac{1}{2} B I_{202}(0) + \frac{3}{4} B I_{202}(2/\sqrt{\pi} B),
\] (P2-5)

\[
\sigma_{^{0200}} = \frac{4}{5} B I_{202}(0) + \frac{11}{4} B I_{202}(2/\sqrt{\pi} B).
\] (P2-6)

Note that Eqs. (P2-4) and (P2-6) are identical with the expressions for these cross sections derived in Ref. 11 by a different, more approximate method. The correspondence is seen when the identifications

\[
BI_{202}(0) = \sigma^{(0)} \quad \text{and} \quad I_{202}(2/\sqrt{\pi} B)/I_{202}(0) = x
\]

are made.
a) Exact DWBA expressions:

\[
\sigma^{(010)}_{(010)}(\omega) = \frac{20}{27} B^2 Q^2 \sum_{j_i, j_f=0}^{\infty} (j_1 + 1)(j_2 + 1) \exp[-B((j_1 + 1)^2 + (j_2 + 1)^2)] \\
\times [I_{112}(2B(j_1 + j_2 + 2)) + (j_1 - j_2)^2 I_{112}(2B|j_1 - j_2|)].
\]

(DD-1)

\[
\sigma^{(0100)}_{(0100)}(\omega) = \frac{5}{3} \left[ Q^2 \langle J^2 \rangle \right]^{-1} \sum_{j_i, j_f=0}^{\infty} (j_1 + 1)(j_2 + 1) \exp[-B((j_1 + 1)^2 + (j_2 + 1)^2)] \\
\times [I_{112}(2B(j_1 + j_2 + 2)) + I_{112}(2B|j_1 - j_2|)].
\]

(DD-2)

b) High temperature expressions:

\[
\sigma^{(010)}_{(010)}(\omega) = \frac{5}{3} B[(4 + \pi) I_{112}(2\sqrt{2}/\pi B) + (4 - \pi) I_{112}(2 - \sqrt{2}/\pi B)].
\]

(DD-4)

\[
\sigma^{(0100)}_{(0100)}(\omega) = \frac{5}{3} B[I_{112}(2 \sqrt{2}/\pi B) + I_{112}(2 - \sqrt{2}/\pi B)].
\]

(DD-5)

\[
\sigma^{(0200)}_{(0200)}(\omega) = \frac{5}{2} B[(1 + \delta^{(\epsilon)} \pi) I_{112}(2\sqrt{2}/\pi B) + (1 - \delta^{(\epsilon)} \pi) I_{112}(2 - \sqrt{2}/\pi B)].
\]

(DD-6)

From the foregoing high temperature expressions, a number of approximate relations between diagonal cross sections may be derived. These are listed in Table II, also for the quadrupole-quadrupole interaction explicit expressions for which have been omitted because of their complexity.

Table 2. Relations between Diagonal Collision Cross Sections (High Temperature Approximation).

<table>
<thead>
<tr>
<th>Potential Symbol</th>
<th>Relation</th>
</tr>
</thead>
<tbody>
<tr>
<td>P1</td>
<td>(\sigma^{(10)}<em>{(01)} \approx \sigma^{(010)}</em>{(01)} \approx \sigma^{(0100)}_{(0100)} )</td>
</tr>
<tr>
<td>P1</td>
<td>(\sigma^{(00)}<em>{(10)} \approx \sigma^{(0100)}</em>{(0100)} )</td>
</tr>
<tr>
<td>P2</td>
<td>(\sigma^{(00)}<em>{(10)} \approx \sigma^{(0100)}</em>{(0100)} )</td>
</tr>
<tr>
<td>DD</td>
<td>(\sigma^{(00)}<em>{(10)} \approx \sigma^{(0100)}</em>{(0100)} )</td>
</tr>
<tr>
<td>QQ</td>
<td>(\sigma^{(00)}<em>{(10)} \approx \sigma^{(0100)}</em>{(0100)} )</td>
</tr>
</tbody>
</table>

4. A Model for the Transition Operator

The program sketched at the end of Section 2 is rather formidable. Therefore it is useful to try and find models for the transition operator which either furnish new relations between collision integrals or make an approximate evaluation of them much simpler. In this section a model transition operator will be described which not only gives a new set of approximate relations, including an approximate law of corresponding states, but may also be specialized (Section 5) to yield rather easily calculable expressions for the contributions to the collision integrals due to multipole-multipole interactions. The starting point of the model is again Eq. (2.6). This expression would be much more useful if the full Born approximation could be used, i.e., if the solutions \(\chi_{k}^{(+)}(R)\) could be replaced by plane waves. It is now assumed that this is the case, provided that the potential \(V_{112L}(R)\) is replaced by another potential, \(\tilde{V}_{112L}(R)\). With this assumption, Eq. (2.6) becomes

\[
\Lambda^{(L)}_{112L}(g, g') = h^{-3} \int dR \exp(iK \cdot R) \tilde{V}_{112L}(R) \times [R]^{(L)},
\]

(4.1)

where \(K = k' - k\). It is furthermore assumed that \(\tilde{V}_{112L}(R)\) has the form

\[
\tilde{V}_{112L}(R) = C_{112L} F_{112L}(R|d_{112L}|),
\]

(4.2)

* It should be noted that Eq. (4.1) is only approximate. There does not in general exist a local \(\tilde{V}_{112L}(R)\) such that Eq. (4.1) is exact. It is, however, hoped that Eq. (4.1) is a reasonable approximation for a large range of values of \(k\) and \(k'\).
where $C_{n L}$ has the dimension of an energy and $d_{n L}$ is a length characteristic of $\hat{V}_{n L}(R)$. Using Eq. (4.2) and the expansion of the plane wave in spherical Bessel functions

$$\exp[i\mathbf{K}\cdot\mathbf{R}] = \sum_{m=0}^{\infty} \frac{(2m+1)!}{2^m m!} \times i^m j_m(k R)[\mathbf{K}]^{(m)} \Theta^m[\mathbf{R}]^{(m)}, \tag{4.3}$$

the result

$$A^{(L)}_{n L}(g, g') = 4\pi \hbar^{-3} l_L C_{n L}$$

$$\times d_{n L}^{(L)}[\mathbf{K}]^{(L)} G_{n L} \hat{K} d_{n L} \tag{4.4}$$

with

$$G_{n L}(x) = \int_0^\infty dy y^2 j_L(x y) F_{n L}(y) \tag{4.5}$$

is obtained. When Eq. (4.4) is inserted into Eq. (2.12) for $\sigma_{n L}(\epsilon)$ and the transformation (2.33) is used, the following result is obtained after some calculation:

$$\sigma_{n L}(\epsilon) = \exp(\epsilon/2) \frac{l}{l} x_{n L}^{(\pm)}(\epsilon)$$

$$\times \frac{1}{3} \left( L + 1 \right) \left( 1 - \delta_{L,0} \right) \tag{4.6}$$

with

$$\sigma_{n L}(\epsilon) = \pi m C_{n L}^2 d_{n L}^2 \left( \hbar^2 k T (2 L + 1)! \right)^{-1} \tag{4.7}$$

and

$$I_{n L}(\epsilon) = \int_0^\infty dy y |G_{n L}(y)|^2$$

$$\times \exp\left[ - \frac{\epsilon^2}{16 \pi^2 y^2} \right], \tag{4.8}$$

where

$$x^2 = \frac{\hbar^2}{4} \left( \frac{d_{n L}^2}{m} k T \right). \tag{4.9}$$

It is obvious from these equations that now all collision integrals may be approximately calculated if $\hat{V}_{n L}(R)$ is given. A specific case will be studied in the next section. The form (4.4), however, gives rise to a general relation between $\sigma_{n L}(\epsilon)$ and the transformation (2.33) used, the following result is obtained after some calculation:

$$\sigma_{n L}(\epsilon) = \exp(\epsilon/2) \frac{l}{l} x_{n L}^{(\pm)}(\epsilon)$$

$$\times \frac{1}{3} \left( L + 1 \right) \left( 1 - \delta_{L,0} \right) \tag{4.6}$$

with

$$\sigma_{n L}^0(\epsilon) = \pi m C_{n L}^2 d_{n L}^2 \left( \hbar^2 k T (2 L + 1)! \right)^{-1} \tag{4.7}$$

and

$$I_{n L}'(\epsilon) = \int_0^\infty dy y |G_{n L}(y)|^2$$

$$\times \exp\left[ - \frac{\epsilon^2}{16 \pi^2 y^2} \right], \tag{4.8}$$

where

$$x^2 = \frac{\hbar^2}{4} \left( \frac{d_{n L}^2}{m} k T \right). \tag{4.9}$$

In order to derive Eq. (4.13), the full form (4.4) is not really necessary, the tensorial dependence on $[\mathbf{K}]^{(L)}$ being crucial only. Such a tensorial dependence may be derived from much less stringent assumptions than those made in the first part of this section. It is therefore reasonable to expect that the relations of Table III are rather accurate. This is reinforced by the fact that the relation for the single-P2 potential, which has been derived earlier, has been checked rather extensively. Another check can be had by combining the relation for the $P_1$ potential in Table III with Eqs. (P1-4) and (P1-6), the result being that $\sigma_{(2000)}^{(0010)} (P_1) \simeq 4.8 \sigma_{(2000)}^{(0000)} (P_1)$. Köhler has performed a quantum mechanical calculation of both $\sigma_{(2000)}^{(0010)}$ and $\sigma_{(2000)}^{(2000)}$ for the $P_1$ potential for HD. He finds $\sigma_{(2000)}^{(0010)} (P_1) = 4.7 \sigma_{(2000)}^{(2000)} (P_1)$, so that the approximation may be said to be excellent in this case.
Table 3. Relations between $\sigma^{(000)}_{(120)}$ and $\sigma^{(010)}_{(101)}$.

<table>
<thead>
<tr>
<th>Potential Symbol</th>
<th>From Eq. (4.14)</th>
<th>High temperature form</th>
</tr>
</thead>
<tbody>
<tr>
<td>P1, P2</td>
<td>$(15)^{1/2} { 10 B \langle F^2 (F^2 - 9/4) \rangle }^{-1}$</td>
<td>$(30)^{1/2}/20$</td>
</tr>
<tr>
<td>DD</td>
<td>$(15)^{1/2} { 20 B \langle F^2 (F^2 - 9/4) \rangle }^{-1}$</td>
<td>$(30)^{1/2}/40$</td>
</tr>
<tr>
<td>QQ</td>
<td>$(15)^{1/2} { 14 B \langle F^2 (F^2 - 9/4) \rangle }^{-1}$</td>
<td>$(30)^{1/2}/28$</td>
</tr>
</tbody>
</table>

The full Born approximation [with $T > l/2l (-??)$], with the results (4.6-9), may be used to derive an approximate law of corresponding states for any collision integral. Consider two gases, distinguished by indices $A$ and $B$. Since all collision integrals depend, in the approximation used, on the temperature only via $B, x$, and $\sigma^{(0)}_{11,2L}$, the equality

$$\langle l_1 l_2 L \mid \sigma^{(p-p)}_{\langle q' q' \rangle k} \mid l_1 l_2 L \rangle^A (T)$$

$$= \langle l_1 l_2 L \mid \sigma^{(p-p)}_{\langle q' q' \rangle k} \mid l_1 l_2 L \rangle^B (T')$$

will be satisfied if $T$ and $T'$ simultaneously satisfy the equations

$$T' = T = \frac{m_A}{m_B}$$

and

$$T' = (m_A/m_B)^2 I_A / I_B$$

and if the function $G_{11,2L}(x)$, or, equivalently, the function $F_{11,2L}(x)$, is the same for both gases. Equations (4.17) and (4.18) imply that the equation

$$I_A m_B (dB_{11,2L})^2 = I_B m_A (dA_{11,2L})^2$$

has to be obeyed. If there exists a choice of values of the $d_{11,2L}$ such that this is the case and if

$$F_{11,2L}(x) = F_{11,2L}(x)$$

for this choice, then Eq. (4.16) may be written as

$$\langle l_1 l_2 L \mid \sigma^{(p-p)}_{\langle q' q' \rangle k} \mid l_1 l_2 L \rangle^A (T)$$

$$= \left[ I_A C_{A1,2L} d^{A1,2L} I_B C_{B1,2L} d^{B1,2L} \right] \langle l_1 l_2 L \mid \sigma^{(p-p)}_{\langle q' q' \rangle k} \mid l_1 l_2 L \rangle^B \cdot (I_A T / I_B).$$

This is the sought-for law of corresponding states.

When the gases $A$ and $B$ are both homonuclear with atoms which are isotopes, then Eqs. (4.19) and (4.20) are certainly valid, since for two such molecules the relations

$$d^{A1,2L} = d^{B1,2L}, \quad C_{A1,2L} = C_{B1,2L}, \quad I_A m_B = I_B m_A,$$

are very closely satisfied. Equation (4.20) simplifies in that case to

$$\langle l_1 l_2 L \mid \sigma^{(p-p)}_{\langle q' q' \rangle k} \mid l_1 l_2 L \rangle^A (T)$$

$$= (m_A/m_B)^2 \langle l_1 l_2 L \mid \sigma^{(p-p)}_{\langle q' q' \rangle k} \mid l_1 l_2 L \rangle^B (m_A T / m_B).$$

If the intermolecular potential is such that no cross terms between different types of potential have to be taken into account for the cross section under consideration, then Eq. (4.22) implies

$$\sigma^{(p-p)}_{\langle q' q' \rangle k}^A (T) = (m_A/m_B)^2 \sigma^{(p-p)}_{\langle q' q' \rangle k} (m_A T / m_B).$$

This formula may be expected to be useful for, e.g., the prediction of the temperature dependence of a cross section for $^{13}N_2$ from the (presumably) measured temperature dependence of the same cross section for $^{14}N_2$. If the two isotopes have different $g(j)$ functions (e.g., $H_2$ and $D_2$), then Eq. (4.22) is not valid (at least not for low temperatures). In such a case the relation

$$I_{A1,2L} (T) = (m_A/m_B)^2 I_{B1,2L} (m_A T / m_B)$$

may still be useful. This is probably especially true for the hydrogen isotopes, for which the inelastic parts of $\sigma^{(000)}_{(120)}$ are negligible.

5. Approximate Calculation of Multipole-Multipole Contributions

In the case of multipole-multipole interactions, the potential $\hat{V}_{11,2L}(R)$ may be approximated using a model due to Chen and Snider. This form is

$$\hat{V}_{11,2L}(R) \begin{cases} 0 & \text{if } R < d, \\ V_{11,2L}(R) = W_{L} R^{-L-1} & \text{if } R \geq d. \end{cases}$$

Here $d$ is a hard-sphere diameter associated with the molecule, which in what follows will be inferred from the expression for $\sigma^{(000)}_{(200)}$ for hard spheres:

$$\sigma^{(200)}_{(200)} = 4/5 \pi d^2. \quad (5.2)$$

$\sigma^{(200)}_{(200)}$ is related to the field-free shear viscosity coefficient $\eta_0$ (see Introduction) It should be noted that this choice of $d$ makes $\hat{V}_{11,2L}(R)$ temperature dependent; this feature of the approximation should improve the temperature dependence of the collision integrals evaluated with this model. On the other hand, the law of corresponding states derived in the previous section does not hold any longer, except for the isotope effect.
The quantity \( W_L \) occurring in Eq. (5.1) may be expressed in terms of the multipole moments of the molecules; in particular,

\[
W_2 = 3 \mu_e^2, \quad W_3 = \frac{15}{4} \mu_e Q_e, \quad W_4 = \frac{105}{4} Q_e^2,
\]

(5.3)

where \( \mu_e \) and \( Q_e \) are the scalar values of the molecular dipole and quadrupole moments, respectively.

With the use of Eq. (5.1), the quantity \( C_{112L} \) and the functions \( F_{112L} \) and \( G_{112L} \) may be calculated to be given by

\[
C_{112L} = W_L d^{L-1},
\]

(5.4)

\[
F_{112L}(x) = \begin{cases} 0 & \text{if } x < 1, \\ x^{-L-1} & \text{if } x \geq 1. 
\end{cases}
\]

(5.5)

\[
G_{112L}(x) = j_{L-1}(x)/x.
\]

(5.6)

With these equations, Eqs. (4.6-9) become

\[
\sigma_{112L}(\epsilon) = \exp(\epsilon/2) I_{112L}(\{\epsilon\}) = \exp(\epsilon/2) \sigma^{(0)}_{112L} J_{112L}(\{\epsilon\}),
\]

(5.7)

\[
\sigma^{(0)}_{112L} = \pi m W_L^2 2L+1 (L)!^2 \\
\times [h^2 k T d^{2L-2}(2L + 1)!]^{-1},
\]

(5.8)

\[
J_{112L}(\{\epsilon\}) = \int_0^\infty dy y^{-1} [j_{L-1}(y)]^2 \\
\times \exp\{-\beta^2 y^2 - \epsilon^2/(16 \beta^2 y^2)\},
\]

(5.9)

\[
\beta^2 = \frac{k^2 (4d^2 m k T)^{-1}}{1/2}.
\]

(5.10)

The integral (5.9) has to be calculated numerically (but see Appendix B for the special case that \( \epsilon = 0 \)).

Once this is done, however, the contributions due to multipole-multipole interactions to the collision integrals of Section 2 may easily be calculated.

6. Summary

The theoretical results of the present article may be summarized as follows:

a) General exact (in the DWBA) expressions for a number of diagonal collision integrals have been derived in terms of a cross section \( \sigma_{112L}(\epsilon) \) which may be calculated once \( V_0(R) \) and \( V_{112L}(R) \) are known. These equations may be found in Section 2.

b) The aforementioned expressions have been reduced for a number of types of nonspherical potentials. From these, approximate expressions have been derived by means of the high temperature approximation. A number of approximate relations between different diagonal cross sections is also given. (Section 3 and Table II).

c) In Section 4 the Born approximation (with a suitably modified \( \tilde{V}_{112L}(R) \)) has been used to derive a relation between \( \sigma^{(0)}_{(0001)} \) and \( \sigma^{(0101)} \) for any type of nonspherical potential. Using the same approximation, a general (approximate) law of corresponding states is found.

d) A model for the transition operator for multipole-multipole interactions has been introduced in Section 5. This model reduces the calculation of collision integrals to one numerical integration for each value of the dimensionless energy transfer \( \epsilon \).

The expressions established in this article may be applied to a large number of problems. Before this can be done, however, much more experimental material has to be available. It would be particularly interesting to see the results of temperature studies of the shear viscosity Senftleben-Beenakker effects for a large number of gases.

Acknowledgements

A large part of the work reported here was done when the author was at the Department of Statistics, University of Waterloo, Waterloo, Canada. Thanks are due to Dr. F. R. McCourt for a number of stimulating discussions.

Appendix A

Evaluation of Traces over Degenerate Internal States

In order to evaluate traces over degenerate internal states the following procedure is used:

a) The trace is rewritten in the \( r^- \)-representation by use of the relation

\[
\langle r^- | P_j | J^{(k)} | r^- \rangle = \frac{\beta_j^{(k)}}{4 \pi (2j - k)!} \int dr^- \frac{dr^-}{dr^-} |(r^-)^{[j]} | (r^-)^{[j]} \rangle
\]

with

\[
\beta_j^{(k)} = \frac{2j(j + 1)!!}{(2j - k)!!} i^k
\]

(5.10)

This equation may be inferred from a slightly different expression given in Reference 12.

In the case of Eq. (2.10) the result of this change of representation is:

\[
\text{tr}_{\text{deg}} \{ P_j | r^- \rangle \langle r^- | J^{(k)} \}
\]

(5.11)

\[
= \beta_{j}^{(0)} \beta_{j}^{(-k)} \int dr^- \frac{dr^-}{dr^-} |(r^-)^{[j]} | (r^-)^{[j]} \rangle

\times |(r^-)^{[j]} | (r^-)^{[j]} \rangle
\]

(5.12)
b) The integrals resulting after step a) are always of the form
\[ \int \mathrm{d}r^j [r^j](j^k)[r^l](l^l) = \mu(j, k, l) T(j, k, l) \]  
(A.4)
where
\[ \mu(j, k, l) = 4 \pi \frac{j! k! l!}{(J + 1)! \left(\frac{1}{2} J - j\right)! \left(\frac{1}{2} J - k\right)! \left(\frac{1}{2} J - l\right)!} \]  
(A.5)
\[ \text{if } J = j + k + l \text{ is even and if the of indices } (j, k, l) \text{ satisfies the triangle inequalities. If one of these two conditions is not met, then } \mu(j, k, l) = 0. \]
This property gives rise to the collisional selection rules of Table I.

Equation (A.3) reduces to the following, when Eq. (A.4) is used:
\[ \text{tr}_{\text{deg}} \{ \mathcal{P}_j [r^j](l^l) \mathcal{P}_j [r^l](l^l) \} = \varphi_j(0) \varphi_j(0) \mu^2(j, l, j') T(j, l, j') T(j, l, j') \]  
(A.6)
where a bar over a digit in a \( (3-j) \)-tensor denotes a contraction of the corresponding set of indices. The result of this contraction may be found by contracting both sides of Eq. (A.6) completely:
\[ \text{tr}_{\text{deg}} \{ \mathcal{P}_j [r^j](l^l) \mathcal{P}_j [r^l](l^l) \} = \varphi_j(0) \varphi_j(0) \mu^2(j, l, j') \Omega(j, l, j') \]  
(A.7)
\[ \text{where } \Omega(j, l, j') = T(j, l, j') T(j, l, j'). \]
\[ \text{The } \Omega(j, k, l) \text{ have been given by Coope}^{15} \text{ as} \]
\[ \Omega(j, k, l) = \left( \frac{J + 1}{2 j} \right) \left( \frac{J - 2 j}{2 k} \right) \left( \frac{J - 2 l}{2 l} \right) \]  
\[ \times \left\{ \begin{array}{ll} 1 & \text{if } J = j + k + l \text{ even}, \\ 2 & \text{if } J = j + k + l \text{ odd}. \end{array} \right. \]  
(A.8)

Appendix B

The Function \( J_{112L}(0) \)

The function \( J_{112L}(0) \) introduced in Eq. (5.9) may be evaluated analytically for energetically elastic collisions, the result being
\[ J_{112L}(0) = \int_0^\infty dy y^{-1} \{ j_{L-1}(y) \}^2 \exp(-\beta^2 y^2) \]  
\[ = \frac{2^{2L-1} \beta^{2L-2}}{\Gamma(L + \frac{1}{2})^2} \frac{1}{2L} \]  
(B.1)
where \( 2F_2 \) is a generalized hypergeometric function. In many cases \( \beta \) is so small that it may be set equal to zero. In this special case the integral is of Weber-Schafheitlin type and has the value
\[ J_{112L}(0) \big|_{\beta=0} = \int_0^\infty dy y^{-1} \{ j_{L-1}(y) \}^2 \]  
\[ = \frac{2L(L-1)}{(2L-1)^{-1}}. \]  
(B.2)

References:

5. H. Moraal and F. R. McCourt, Physica 60, 73 [1972].