Relativistic Doppler Effect and Group Velocity in Homogeneous 
Nonlinear and Inhomogeneous Plasmas 

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(Z. Naturforsch. 26 a, 1531—1538 [1971]; received 30 April 1971)

We study the variations of the received frequencies (Complex Doppler effect) from a source moving uniformly in a homogeneous isotropic plasma and emitting a uniform EM plane wave with amplitude $E_0$, as a function of $\omega_0/E_0$ where $\omega_0$ is the plasma frequency. The ratio of the velocity $U$ of the source over the group velocity $U_J$ of the transmitted wave is then calculated from the slope of the Doppler curves.

When the plasma is inhomogeneous (one-dimensional stratification) the Doppler effect depends on the particular profile of the refractive index and its higher (even) derivatives with respect to the direction of stratification $z$. These new branches are separated from the "old" ones by cut-off regions. The two branches for $\beta > 0$ and $\beta < 0$ of the transmitted frequency $\omega_0$ in the rest frame of the source vs. $\omega_0$ for given values of $\omega_0/E$. The relation between $\omega_0$ and $\omega$ is

$$\omega_0 = \frac{\omega}{\sqrt{1 - \beta^2}} \left[1 - \beta n(\omega)\right] \tag{2}$$

with $\beta = U/c$ and for motion of the source parallel to the direction of the wave number of the transmitted wave. Expression (2) above is deduced by applying the phase invariance method (under Lorentz transformations) to the phase of the traveling plane wave. $E$ is evaluated at the observer's frame.

In the case $E \to 0$ it is known that for a receding source ($\beta < 0$) one frequency $\omega$ is received for a given $\omega_0$ and for an approaching source ($\beta > 0$) two frequencies $\omega$ are received for any $\omega_0$ between $\omega_0/\sqrt{1 - \beta^2}$ and $\omega_0$. As the sequence of Figs. 1 shows for $E \to 0$, new branches of the Doppler effect emerge from the origin — both for $\beta > 0$ and $\beta < 0$. These new branches are separated from the "old" ones by cut-off regions. The two branches for each $\beta$ approach as $\omega_0/E$ decreases, and for $\omega_0/E \to 1.65$ they merge. As $E/\omega_0$ increases further we have again one branch — starting now from the origin — for each $\beta$ and as $E/\omega_0 \to \infty$, $n(\omega) \to 1$ (the plasma is squeezed away from the source to infinity) and one gets $\omega_0 \to \omega \cdot \sqrt{(1 - \beta)/(1 + \beta)}$, i.e. the vacuum case.

A. Doppler Shift in Nonlinear Homogeneous Media

The relativistic Doppler effect in a homogeneous isotropic plasma has been studied so far 1 under the assumption of zero-field-intensity for the EM wave transmitted from the moving source. This means that in the evaluation of the refractive index of the medium the Electromagnetic pressure has been altogether omitted vs. the thermal pressure of the gas or, the plasma has been considered at zero temperature [refractive index $n(\omega) = \left\{1 - \omega_0^2/\omega^2\right\}^{1/2}$]. In any realistic case however, involving the motion of EM sources in dispersive media it is obvious that the intensity of the transmitted waves should be taken into account. It has been proved 2 that such a biasing Electric field $E_0 = E(\tilde{r})$ modifies the cold-plasma refractive index as following:

$$n(\omega) = \left\{1 - \left(\omega_0^2/\omega^2\right) e^{-E_0 T_0}\right\}^{1/2} \tag{1}$$

where $E = E_0 \sqrt{|B|} n k T$, $e$, $m$ are the electron charge and mass, $k$ is the Boltzmann's constant and $T$ is the plasma temperature. Eventually Eq. (1) describes a non-homogeneous medium and for such a medium the principle of phase invariance from which the Doppler shift is deduced 1, is not valid. However, far from the source the field can be considered quasi-plane, $E$ may be taken as constant: consequently $n(\omega)$ in Eq. (1) is taken as an $r$-independent function.

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The interesting feature with the above curves is that now for a given value of $\omega_0$ there correspond two values of $\omega$ to any $\beta$ positive or negative and that near the critical region of coalition we receive three distinct frequencies.

From (2) we get

$$\frac{d\omega_0}{d\omega} = \frac{1}{\sqrt{1-\beta^2}} \left[ 1 - \beta n(\omega) \right] - \frac{\beta \omega}{\sqrt{1-\beta^2}} \frac{dn(\omega)}{d\omega} =$$

$$= \frac{1}{\sqrt{1-\beta^2}} \left( 1 - \frac{U}{U_{cr}} \right)$$

(3)
Here, it has been implicitly assumed that the group velocity in a non-linear medium is given by the same expression as in the usual case of a linear medium, i.e. $U_{gr} = \frac{\partial \omega}{\partial K} / \sqrt{\partial K \partial \omega(K,0)/\partial K}$. In general, for highly non-linear media, i.e. for strong fields $E$ this assumption is not correct since from the dispersion relation $\omega = f(K, E^2)$ one gets:

$$\omega \approx \omega_0(K) + \left( \frac{\partial^2 \omega}{\partial E^2} \right)_0 E^2 + \text{(higher-order terms)}$$

where $\omega_0(K)$ determines the dispersion law in the linear approximation. (For a lucid discussion see Ref. 6.) However, in our case, the variable is the ratio $\omega_0/E$ (from 0 to $\infty$) and small values of this variable do not necessarily imply large values for the normalized field $E$ (see note added in proof at the end of the text).

So, the ratio $Y = U/U_{gr}$ can be evaluated from the curves of Figs. 1. Direct calculation of the group velocity

$$U_{gr} \approx \frac{\partial \omega}{3K} = \frac{c}{1-(\omega_0^2/\omega_0^2)} \frac{e^{-E^2/\omega_0^2}}{e^{-E^2/\omega_0^2}}$$

shows that, due to the very steep variation of $n(\omega)$ with $\omega$, the above expression does not represent the velocity of energy transport beyond a certain limit for which $U_{gr}$ exceeds $c$. This happens because in this case higher order derivatives of $\omega$ with respect...
to the wavenumber $K$ cannot be omitted. A first-order correction intending to take into account a few higher derivatives $\mathcal{E}^2/\mathcal{E}K^2$, $\mathcal{E}^3/\mathcal{E}K^3$ (l.c. 4) gives for the present case:

$$U_{gr} \approx \frac{24}{\tau} \left[ 1 - \frac{\omega e^2}{\omega^2} e^{-E/\omega} \right] \left[ 3 - \frac{27 E^2}{\omega^2} + 24 \frac{E^4}{\omega^4} - 4 \frac{E^6}{\omega^6} \right]$$

where $\tau \approx 10^{-23}$ sec is the natural line width constant. This correction prevents $U_{gr}$ from acquiring infinite values, but otherwise gives poor results for a particularly steep profile $n(\omega)$ such as the one considered here.

It appears that in such cases a "velocity of energy-transport" cannot be defined unambiguously. In Figs. 2 the plots of $U/U_{gr}$ deduced from Eq. (3) are displayed in full line in the allowed regions, where $U_{gr}$ can be defined in the conventional way.

i.e. for $|U/U_{gr}| > |\beta|$. For $\omega_e/E$ below the critical value $\sim 1.65$, $U_{gr}$ undergoes a single minimum at frequencies $\omega < \omega_e$. As $\omega_e/E$ increases beyond this critical value, cut-off regions appear and the shape of the curves changes: There appears now a peak of $U/U_{gr}$ at frequencies $\omega < \omega_e$ approaching $\omega_e$

and finally for $E \to 0$, 
\[
\frac{d(U/U_{gr})}{d\omega} = \frac{-\beta (\omega_e^2/\omega^2)}{[1 - (\omega_e^2/\omega^2)]^{3/2}} 
\]
i.e. $U/U_{gr} \to \infty$ or $U_{gr} \to 0$, for $\omega \to \omega_e$.

For all cases $U/U_{gr} \to \beta$ as $\omega \to \infty$. It is now of some interest to evaluate the group velocity of the transmitted wave in the rest frame of the source, i.e. as seen from an observer moving with the source. Denoting by primes the parameters in the rest frame of the medium we write for the wave number and the frequency, the transformation relations:
\[
\begin{align*}
K' &= K - \gamma \frac{\omega}{c^2} U + \frac{K \cdot U}{UT} U(\gamma - 1), \quad \gamma = \frac{1}{\sqrt{1 - \beta^2}}, \\
\omega' &= \gamma(\omega - K \cdot U) 
\end{align*} \tag{4}
\]
and \[
U_{gr}' = \frac{\partial \omega}{\partial K} = \nabla_K(\omega) \text{ i.e. } U_{gr,x,y} = \frac{\partial \omega}{\partial K_x, y}. \tag{5}
\]

We assume $K$ in the $x, y$ plane, forming an angle $\Theta$ with the $X$ direction; $U$ is taken along the $X$ axis. From Eqs. (4), (5) we get:
\[
K_x' = K_x - \gamma \frac{\omega}{c} U + K_x(\gamma - 1) = \gamma[K_x - \beta(\omega/c)],
\]
\[
K_y' = K_y, \quad K_z' = K_z,
\]
and \[
\omega' = \gamma(\omega - K \cdot U).
\]

So, \[
U_{gr}' = \frac{\partial \omega'}{\partial K_x} = \frac{1}{(\partial K_x/\partial \omega)(d\omega/d\omega')} = \frac{d\omega'/d\omega}{\partial K_x/\partial \omega'},
\]
but \( \frac{d\omega'}{d\omega} = \gamma (1 - \frac{U}{U_{grz}}) \),
\[ \frac{dK_x'}{d\omega} = \gamma \frac{1}{(1 - \frac{U}{U_{grz}} - \beta/c)} \]
so,
\[ U_{grz} = \frac{U_{grz} - U}{1 - (1 - \frac{U}{U_{grz}} - \beta/c)} \quad \text{and} \quad U_{grz} = \frac{U_{grz} + U}{1 + (1 - \frac{U}{U_{grz}} - \beta/c)} \tag{7} \]

i.e. we get the usual expression expected from the Lorentz transformations of the velocity.

### B. Doppler Shift in Inhomogeneous Media

In a (mildly) non-homogeneous medium (one-dimensional stratification), the Doppler shift can be calculated in an explicit way only for \( U \ll c \). For motion parallel to the direction of stratification one obtains

\[ \left| \frac{\Delta \omega}{\omega} \right| \approx U \sum_{m = 0}^{\infty} (-1)^m \frac{1}{(2K)^{2m}} \frac{d^{2m} K(z)}{dz^{2m}} \tag{10} \]

where \( \left| \frac{\Delta \omega}{\omega} \right| = \left| \omega_0' - \omega \right| \) and \( K(z) \) is the wavenumber as a function of “height”. In cases where the series (10) converges, \( K \gg 1 \) is a desirable prerequisite for fast convergence.

There exist two obvious cases where (10) can be evaluated exactly: a) the sinusoidal and the b) exponential profile.

a) For a harmonic profile we write

\[ K = \frac{\omega}{c} \sqrt{e^2} = A + \sin z > 1 \]

where \( A \) is a constant and \( \varphi = f(\omega) \).

Then,
\[ \frac{d^{2m} K}{dz^{2m}} = (-1)^m \omega^{2m} \sin z \quad \text{so}, \quad \frac{\Delta \omega}{\omega} = \sin(q z) \sum_{m = 0}^{\infty} \frac{(-1)^m}{(2K)^{2m}} \frac{d^{2m} K(z)}{dz^{2m}} \]

where

\[ \lambda = \frac{q}{2[A + \sin(q z)]} = \frac{q}{2K} \]

under the condition \( \lambda < 1 \).

Now \( \frac{\Delta \omega}{\omega} = \frac{K}{\omega} \left[ \frac{dK}{\omega} \left( 1 - \lambda^2 \right) - \frac{(K - A) 2 \lambda}{d\omega} \right] \)

\[ = \frac{U}{U_{grz} (1 - \lambda^2) - 2 \lambda (K - A) \frac{d\omega}{d\omega}} \]

\[ = Y (1 - \lambda^2) + \lambda (K - A) \frac{\varphi \left( \frac{dK}{\omega} \right) - \frac{K}{\omega}}{K^2} \]

and since

\[ \frac{\Delta K}{\omega} = q \cos(q z) \frac{2\varphi}{2\omega}, \quad \frac{\Delta \varphi}{\omega} = \frac{\Delta K}{\omega} \frac{1}{q \cos(q z)} \]

we get finally

\[ \frac{\Delta \omega}{\omega} = Y \frac{(1 - \lambda^2) + \lambda (K - A) \frac{\varphi \left( \frac{dK}{\omega} \right) - \frac{K}{\omega}}{K^2}}{(1 - \lambda^2)^2} \tag{11} \]

from which \( Y \) is deduced as a function of the slope of the Doppler shift.

b) For an exponential profile, \( K = (\omega/c) e^{-\gamma z} \), where \( \gamma \) is a constant so,

\[ \frac{\Delta \omega}{\omega} = \frac{\omega}{c} e^{-\gamma z} \sum_{m = 0}^{\infty} (-1)^m \frac{\gamma^{2m} (2K)^{2m}}{(2K)^{2m}} \]

and

\[ \left| \frac{\Delta \omega}{\omega} \right| < 1. \]

Further,

\[ \frac{\Delta \omega}{\omega} = \frac{\omega}{c} e^{-\gamma z} \sum_{m = 0}^{\infty} (-1)^m \frac{\gamma^{2m}}{(2K)^{2m}} Y \]

In the general case the relation (10) rarely converges when \( \left| K(\omega, z) \right| < 1 \). For a linear profile one gets

\[ \frac{\Delta \omega}{\omega} = \frac{\omega}{c} e^{-\gamma z} \sum_{m = 0}^{\infty} (-1)^m \frac{\gamma^{2m}}{(2K)^{2m}} Y \]

Differentiation of both sides of (10) with respect to \( \omega \) gives:

\[ \frac{d^2 Y}{dz^2} = \frac{4 K^2 + 2 d^2 K}{K \frac{d^2 K}{dz^2}} \]

and for \( K \sim a z^2 \) the above becomes

\[ \frac{d^2 Y}{dz^2} = \frac{4 a^2 z^4 + \frac{4}{z^2}}{Y} \]

In Figs. 3 a, b and 4 a, b we display \( \left| \frac{\Delta \omega}{\omega} \right|/|U| \) vs. \( \omega_c/\omega \) and \( \left| \frac{\Delta \omega}{\omega} \right|/|U| \) vs. \( z \) for the cases of a symmetric

\[ \left( X = \frac{\omega_s^2}{\omega^2} = \frac{\omega_s^2}{\omega^2} \left( 1 + e^2 \right) \right) \]

and a transition ionospheric layer

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in which in addition a small-scale stochastic variation of the electron distribution \( \Delta N/N \) has been superimposed.

The expression for the “average” wave number for such an irregular slightly (cold) ionized gas has been developed recently

regularities it is given as:

\[ K = \frac{a}{c} \left( 1 - X \left[ 1 + \frac{\frac{a}{2} X \cdot 1}{1 + \frac{1}{2} X} \right] \right)^{\frac{1}{2}} \]

where \( a = \left( \frac{\Delta N}{N} \right)^2 \).

In the present case we have taken \( a = 0.01 \) and \( \omega_{0e} = 10^8 \) Hz. The numerical calculations have been extended up to \( m = 10 \) (tenth derivative). In the study of the variation of \( |\Delta \omega|/U \) with respect to \( \omega_{0e}/\omega \), beyond a certain limit (\( \omega_{0e}/\omega \sim 0.8 \)) \( K(\omega, z) \) becomes \(<1\) so that \( 1/(2K)^m \gg 1 \) and the series (10) beyond that limit oscillates rapidly even for small values of the corresponding derivatives with
respect to $z$. We have plotted in Figs. 3 a, 4 a the corresponding curves within the frequency range for which the series converges rather rapidly ($K > 1$).

The variations of $|\Delta \omega|/U$ with respect to $z$ are more pronounced near the middle of the layers for both the symmetric and the transitional case. For

$\omega = f(K, E^2)$ is the pertinent non-linear dispersion relation. — In our problem however, the transmitted E.M. wave is excited from a fixed source i.e. a source independent of the medium. Consequently (since Maxwell's equations are linear!) the frequency of the E.M. waves is necessarily amplitude independent and the used formula for the group velocity in the text is rigorous. What is "split" here, therefore, is the group velocity of the plasma (longitudinal) waves which are excited by the strong electromagnetic field — a case of no concern here. — It would be of course otherwise if the transmitted E.M. wave is excited from the (non-linear) motion of the plasma particles themselves.

Acknowledgements
The authors thank Dr. E. K. Yfantis for helpful discussions.