HEAT AND DIFFUSION FLUXES IN A IONIZED GAS

Nachweises der ESR 30, 33, 34 wurde im vorliegenden
System gezeigt, daß die \( \tau_{y} \)-Komponente strahlt 18. Dieser Befund stimmt mit dem experimentellen Er­gebnis \( k_2 > k_\parallel > k_3 \) überein, kann dieses aber nicht quantitativ erklären, denn der strahlunglose Anteil der Desaktivierung beträgt etwa 90% 5, 27. Daraus folgt, daß auch für die strahlunglosen Übergänge, die über hohe Schwingungsanregungen des elektrischen Grundzustandes \( \text{A}_g \) erfolgen, Auswahrläger existieren:

\[
k_{\text{ISC}} > k_{\text{IC}} > k_{\text{SC}} = 0.
\]

Diese Auswahlregeln wurden bislang nicht theore­tisch behandelt.

Für ein beliebiges System läßt sich der genaue Weg der Be- und Entvölkerung des phosphoreszier­renden Triplettzustandes nur vorhersagen, wenn


nachten der Symmetrie der Elektronenzustände die energetische Lage aller Anregungsniveaus und die Quat­ten der Molekülschwingungen bekannt sind.

Um die vorliegenden Probleme zu klären, haben wir inzwischen begonnen, weitere Systeme zu unter­suchen. Es zeigte sich, daß alle Systeme mit hinrei­chend „langer“ Spin-Gitter-Relaxationszeit Spin­Polarisationseffekte aufwiesen. Das überraschendste Ergebnis war dabei, daß auch bei Trip­lett-Excitonen

in reinen Naphthalinkristallen bei etwa 300 °K ein \( Jm = 1 \)-Übergang bei spezieller Orientierung emis­siv, und einer absorptiv ist 23.

Herrn Professor Dr. H. C. WOLF danken wir auf­richtig für die hervorragenden Arbeitsmöglichkeiten im 3. Physikalischen Institut und für die stetige Förder­ung dieser Arbeit. Die Arbeit wurde von der Deutschen Forschungsgemeinschaft mit Sachmitteln unterstützt.


Heat and Diffusion Fluxes in a Multicomponent Ionized Gas in a Magnetic Field

I. General Expressions

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(Z. Naturforsch. 24 a, 967—976 [1969]; received 27 February 1969)

Ordinary and thermal diffusion as well as heat flux in a dilute, ionized, multicomponent mono­atomic gas in a magnetic field are considered with the Chapman-Enskog-Burnett method. It is shown how, with certain modifications, the usual expressions for the properties of an un-ionized monatomic gas may be applied to this case. The expression for the diffusion flux is compared with the momentum equation suggested by SCHLITZ.

The importance of transport properties in deter­mining the behaviour of an ionized gas has stimulat­ed many theoretical studies of these properties, most often with the Chapman-Enskog-Burnett method 1. Some expressions for the properties in electric and magnetic fields were given in the first edition of the book by CHAPMAN and COWLING 1, and further work was done by COWLING in 1945 2, but the first accurate expressions for the electron properties of

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3 R. LANDSHOFF, Phys. Rev. 76 [1949]: 82, 442 [1951].
4 L. SPITZER, Jr. and R. HÄRM, Phys. Rev. 89, 977 [1953].
6 S. I. BRAGINSKII, Soviet Phys.-JETP 6, 358 [1958].
combined magnetic and alternating electric fields have been given.

The inclusion of neutral species into accurate expressions for the transport properties has been effected only recently. Cowling included neutral species in the first approximation to the electrical conductivity in a magnetic field. This approximation is, however, highly inaccurate for the strongly ionized gas (\(\geq 1\%\) ionization), and predicts much too small a coefficient at low ionization for certain types of electron-neutral cross sections. Further work included development of expressions for electron coefficients in a three-component plasma without and with a magnetic field. Explicit expressions are now available for up to the sixth approximation to all electron coefficients in complex mixtures in a magnetic field in which the electron-neutral cross sections have an arbitrary dependence on relative energy. The behaviour of the ion and neutral species in a multicomponent plasma without B-field have also been considered.

The purpose of the present work is to extend the expressions for the transport coefficients of a mixture of an arbitrary number of species, both charged and uncharged, to the case where a magnetic field is present. These expressions turn out to be only slightly different from those previously derived for this gas in the absence of a magnetic field. With the electron properties in a magnetic field, it is necessary to introduce complex terms containing the various cyclotron frequencies, and then to extract the real and complex parts of the coefficients. The former is the transverse component, perpendicular to the magnetic field \(B\) and the driving force under consideration \((E, \nabla p_n, \text{etc.})\) and the latter is the so-called Hall component-perpendicular to both \(B\) and to the forcing term. The coefficients parallel to \(B\) are the limit of the complex expressions when \(|B| \to 0\).

In the expressions all species are considered on an equal basis, i.e. the simplifications possible because of the small electron mass have not been introduced. As a result they are also applicable to gases composed of positive and negative ions. More often, the gas under consideration will contain the very much lighter electrons in a concentration equal to or greater than that of the ions. In this case the electron properties will be affected by a much smaller magnetic field than that necessary to influence the ions. Since it is possible to develop much simplified expressions for the electron properties in multicomponent gases, it is possible in this case to compute the influence of the magnetic field without using the more complicated expressions developed here. In an increasing number of experiments, e.g. magnetically confined electric arcs, the magnetic field is so large that the electron coefficient is reduced below that of the ions, and the latter itself is reduced by the magnetic field. In this case, the expressions presented here are useful in predicting the coefficients in regions of partial ionization, or where ions of several sorts are present. Schlüter and, slightly later, Johnson have suggested that a type of momentum equation for each species be used to predict the motion of individual species in an ionized gas. Since these expressions have found wide application in understanding the behaviour of electric arcs, it is worthwhile to see if they can be justified from kinetic theory. A comparison of the results of kinetic theory with the expressions of Schlüter is also given here.

13 The level of approximation refers to the number of terms in the Sonine polynomial expansion, see Eqs. (16) and (29). Some authors number the approximations from the first non-vanishing approximation, so that the first approximation to thermal conductivity and thermal diffusion coefficient are computed with two terms in the expansion.
17 R. S. Devoto, Phys. Fluids 10, 2105 [1967].
18 C. P. Li and R. S. Devoto, Phys. Fluids 11, 448 [1968].
19 R. S. Devoto, Phys. Fluids 9, 1230 [1966].
20 R. S. Devoto, Phys. Fluids 10, 2704 [1967].
Solution of the Linearized Boltzmann Equation

The solution of the Boltzmann equation follows along the same lines as given for the multicomponent gas without magnetic field and for certain properties of the binary ionized gas in a magnetic field. Since the theoretical basis of the derivation is discussed at length in these two references, it will not be repeated here. Only enough of the steps will be given to establish connection with the derivation of the properties of the un-ionized multicomponent gas.

The properties are all given in terms of integrals over the perturbation $\varphi_i$ to the equilibrium Maxwellian distribution $f_i^{(0)}$, which is found from the coupled equations (e.g. s. units) \(^{1,21}\)

$$f_i^{(0)} = \frac{n}{n_i} V_i \cdot d_i + 2 B_i : \nabla v_0 - (\frac{1}{2} - W_i^2) V_i \cdot \nabla \ln T$$

$$- f_i^{(0)} = \frac{m_i}{q_c k T} V_i \cdot (j \times B) + e_i \frac{\partial \varphi_i}{\partial \varphi_i} \cdot (V_i \times B)$$

$$- n_i \sum \delta_{ij} \delta_{ij}(\varphi_i + \varphi_j)$$

$$= \sum \delta_{ij} \delta_{ij}(\varphi_i + \varphi_j) g \sigma \, d\Omega \, d\nu_j \quad (1)$$

with

$$n_i n_j l_{ij}(\varphi_i) = \int \int f_i^{(0)} f_j^{(0)} (\varphi_i - \varphi_j) g \sigma \, d\Omega \, d\nu_j \quad (2)$$

and

$$p d_i = \nabla p_i - \nabla p \cdot \left( \frac{e_i - q_c}{m_i} \right) (E + \frac{v_0 \times B}{c})$$

$$- \frac{1}{q} \left[ \frac{q}{m_i} \frac{X_i}{X_i} - \frac{1}{n_k} X_k \right].$$

$$j$$ is the electric current, $e_i$ is the charge on the $i$th species, $B$ is the magnetic field, $q_c = \sum e_k n_k$ is the charge density (negligible in most gases), and $X_i$ represents all external forces not electromagnetic in nature. Other symbols are those used in Ref. \(^{21}\). When not otherwise indicated, the sums run over all species i.e. from 1 to $v$. For later purposes it is worth noting that the thermal velocity $V_i$ is related to the molecular velocity $v_i$ and the mass average velocity $v_0$ by $V_i = v_i - v_0$.

Magnetic interactions as well as collisions are considered on an equal basis in Eq. (1); situations in which the magnetic effects are dominant as well as those in which collisions are dominant can be treated with the same theory. Since the masses and charges of the several components in, for example, a partially ionized plasma can be considerably different, it is clear that collisions may govern the transport behaviour of certain species, while finite gyroradius effects govern that of others.

It should be noted that the Boltzmann form of the collision integral used here Eq. (2) is, strictly speaking, valid neither for the case of charged-particle interactions, nor in very high magnetic fields, when the field affects the collisions. In the former case, it is well-known that the Boltzmann term does correctly describe charged-particle interactions to dominant order, i.e. large Coulomb logarithm, when either the Coulomb potential with cut-off at the Debye length or the Debye-shielded Coulomb potential is used. The incorporation of results from accurate collision terms \(^{10,11,29}\) into expressions derived with the Boltzmann form is straightforward \(^{30}\).

It should be possible to do the same with collision integrals including the effect of a finite Larmour radius \(^{31,32}\). Under a very wide range of conditions such effects may be ignored \(^{33}\).

The term involving $b_i$ in Eq. (1) contributes only to the stress tensor and, for the purposes of this paper, can be ignored. The solution to the remainder of Eq. (1) can be written in the form

$$\varphi_i = - A_{i||} \cdot \nabla \ln T - A_{i\perp} \cdot \nabla \ln T$$

$$- A_{iH} \cdot \hat{B} \times \nabla \perp \ln T$$

$$+ n \sum \delta_{ij} \delta_{ij}(C_{ij}^|| + C_{ij}^\perp + C_{ij}^{H||} \hat{B} \cdot d_j)$$

\(^{34}\)

with $A_{i||} = A_{i||} W_i$, $A_{i\perp} = A_{i\perp} W_i$, etc. The scalars $A_{i\perp}$, $A_{i\perp}$, etc. are functions of $W_i^2$ and $B^2$, and of the thermodynamic state of the mixture, which in the Chapman-Enskog theory is assumed given by the fluid dynamical equations. $\hat{B}$ is the unit vector $B/B$ and the superscripts $||$, $\perp$, and $^H$ denote the components parallel to $B$ (e.g. $\hat{B} \cdot \nabla T$), perpendicular to $B$ but parallel to the driving force e.g.

$$\nabla T - \hat{B} (\hat{B} \cdot \nabla T) = - \hat{B} \times (\hat{B} \times \nabla T),$$

and in the Hall direction (e.g. $\hat{B} \times \nabla \perp T$). The form chosen here for $\varphi_i$ may be shown via vector manipulation to be the same as that given elsewhere \(^{1-5}\). With this form for $\varphi_i$ the electric current


can be written as
\[ J = \sum_i e_i n_i (V_i) \]
\[ = -M^{\parallel} \nabla^{\parallel} \ln T - M^{\perp} \nabla^{\perp} \ln T - M^{\parallel} \hat{B} \times \nabla^{\perp} \ln T + n \sum_j (L^{||} \mathbf{d}_j^{\parallel} + L^{\perp} \mathbf{d}_j^{\perp} + L^{\parallel} \hat{B} \times \mathbf{d}_j^{\perp}), \]
where the coefficients are given by
\[ M^{\parallel} = \sum \frac{1}{2} e_i (m_i / 2 k T)^{1/2} \int f_i^{(0)} A_i V_i \cdot \mathbf{d} V_i, \]
\[ L^{||} = \sum \frac{1}{2} e_i (m_i / 2 k T)^{1/2} \int f_i^{(0)} C_i^{||} V_i^2 \cdot \mathbf{d} V_i, \]
\[ L^{\perp} = \sum \frac{1}{2} e_i (m_i / 2 k T)^{1/2} \int f_i^{(0)} C_i^{\perp} V_i^2 \cdot \mathbf{d} V_i, \]
and similar relations for the other directions.

The solution to Eq. (1) is linear in the various gradients, which are determined by the equations of magnetohydrodynamics. The gradients may be considered as independent parameters in Eq. (1), and the coefficient of each must vanish. We need consider only the solutions for \( C_i^{||}, C_i^{\perp} \) and \( C_i^{\parallel H} \), since the solutions for \( A_i^{||}, A_i^{\perp} \) and \( A_i^{\parallel H} \) proceed via almost identical steps. From Eqs. (4) – (7), neglecting terms involving \( \nabla^{\parallel} \ln T \) and \( \nabla^{\perp} \ln T \), we obtain
\[ \partial \mathbf{V}_i \cdot \mathbf{B} = -n B \sum_j (C_i^{||} \cdot \mathbf{d}_j^{\parallel} - C_i^{\parallel H} \cdot \hat{B} \times \mathbf{d}_j^{\perp}) \]
and
\[ \mathbf{j} \times \mathbf{B} = n B \sum_j (L_i^{||} \mathbf{d}_j^{\parallel} - L_i^{\perp} \hat{B} \times \mathbf{d}_j^{\perp}). \]
As could be anticipated, we see that \( C_i^{||} \) does not occur in either Eq. (8) or (9). The equation for this component therefore does not contain the magnetic field, and is identical to that obtained in the absence of this field. The solution for \( C_i^{\parallel H} \) and the corresponding transport coefficients parallel to the magnetic field are just those already given for the multicomponent gas\(^{21, 34}\). Equivalent remarks apply to \( A_i^{||} \).

Upon substituting Eqs. (8) and (9) into Eq. (1), making use of the identity \( \sum_j \mathbf{d}_j = 0 \)\(^{21}\), and equating coefficients of \( \mathbf{d}_j^{\perp} \) and \( \hat{B} \times \mathbf{d}_j^{\perp} \), we obtain
\[ 1 \sum_j f_i^{(0)} (\delta_{ih} - \delta_{ik}) \mathbf{V}_i \]
and
\[ \mathbf{j} \times \mathbf{B} = n B \sum_j (L_i^{||} \mathbf{d}_j^{\parallel} - L_i^{\perp} \hat{B} \times \mathbf{d}_j^{\perp}). \]

The finite expansion
\[ C_i^h - C_i^k = \sum_{p=0}^{\xi-1} \phi_i^{(p)} (W_i^2)^{p} \]
where \( S_i^{(p)} (W_i^2) \) are Sonine polynomials\(^{1, 3, 21}\) and the coefficients \( \phi_i^{(p)} \) are, of course, complex. Now form the inner product of \( (2 \pi m_i / k T)^{1/2} S_i^{(p)} (W_i^2) \mathbf{W}_i \) with the resultant equation, integrate over \( \mathbf{d} V_i \) and rearrange to get
\[ \sum_j \sum_{p=0}^{\xi-1} \phi_i^{(p)} (W_i^2)^{p} = 3 \pi^{1/2} \delta_{i0} (\delta_{ih} - \delta_{ik}). \]

All coefficients in the absence of a magnetic field will be denoted by the superscript \( I I \). Symbols without this superscript in the present paper always denote complex combinations of perpendicular (\( \perp \)) and Hall (\( \parallel H \)) coefficients, e. g. \( C_i^h = C_i^{1h} + i C_i^{2h} \).
The elements $q_{ij}^{mp}$ are related to $Q_{ij}^{mp}$ elements introduced in Hirschfelder et al. \textsuperscript{21} by
\begin{equation}
q_{ij}^{mp} = (2 \pi m_i / k T)^{1/2} Q_{ij}^{mp}
\end{equation}
and given in terms of the bracket integrals\textsuperscript{36} by

\begin{equation}
\sum_j m_j \left( \frac{m_j}{2 k T} \right)^{1/2} \int f_j^{(0)} (C_j^h - C_j^l) V_j^2 dV_j = 0 .
\end{equation}

Substituting the expansion of Eq. (16) we obtain
\begin{equation}
\sum_j \bar{q}_{ij}^{mp} = 0.
\end{equation}

This equation is then multiplied by $q_{0i}^{0m}$, divided by $n_i m_i^{1/2}$ and subtracted from Eq. (17) to yield the independent equations
\begin{equation}
\sum_j \tilde{q}_{ij}^{mp} = 3 \pi^{1/2} (\delta_{ik} - \delta_{ij}) (m = 0, 1, \ldots, \tilde{z} - 1)
\end{equation}
with
\begin{equation}
\tilde{q}_{ij}^{mp} = q_{ij}^{mp} - n_i \left( \frac{m_i}{m_j} \right)^{1/2} \sum_k \bar{q}_{ik}^{mp} \delta_{m0} \delta_{p0} .
\end{equation}

Some particular forms of Eq. (28) are:
\begin{equation}
\tilde{q}_{ii}^{00} = 0 ,
\end{equation}
\begin{equation}
\tilde{q}_{ij}^{00} = -8 n_i n_j \left( \frac{m_i}{m_i + m_j} \right)^{1/2} \bar{Q}_{ijkl}^{(1,1)}
- 8 \left( \frac{m_i}{m_i} \right)^{1/2} n_j \sum_k \left( \frac{m_k}{m_i + m_k} \right) \bar{Q}_{ikl}^{(1,1)}
- 3 n_j \left( \frac{2 \pi m_i}{k T} \right)^{1/2} n_i m_i \left( \omega_j - \omega_i \right) \omega_i \right)
\end{equation}
\begin{equation}
\tilde{q}_{ii}^{mm} = \tilde{q}_{ii}^{mm} + i \omega_i n_i \left( \frac{2 \pi m_i}{k T} \right)^{1/2} 2 (m + \frac{3}{2}) ! \frac{1}{\pi^{1/2} m !} (m > 0),
\end{equation}
\begin{equation}
\tilde{q}_{ij}^{mm} = q_{ij}^{mm} (m > 0, i \neq j),
\end{equation}
\begin{equation}
\tilde{q}_{ij}^{mp} = q_{ij}^{mp} (m = p, m > 0).
\end{equation}

We see that the magnetic field affects all elements for $m = p = 0$, but occurs otherwise only in diagonal elements. Thus only minor additions to the usual

\begin{equation}
q_{ij}^{mp} = q_{ij}^{mp} + i \omega_i n_i \left( \frac{2 \pi m_i}{k T} \right)^{1/2} 2 (m + \frac{3}{2}) ! \frac{1}{\pi^{1/2} m !} (m > 0),
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\end{equation}

The proof of this equality follows from two properties of the bracket integrals,
\begin{equation}
[ (0), i; (p), j ] = 0
\end{equation}
and
\begin{equation}
m_i^{1/2} [ (0), i; (p), j ] = - m_j^{1/2} [ (0), j; (p), i ]
\end{equation}
where obvious symbolic notation has been used for the bracket integrals of Eqs. (20) and (21). These properties can be deduced from relations given in Chap. 9 of Ref. \textsuperscript{1}. The redundancy of Eqs. (17) for $m = 5$ can be removed with the aid of the auxiliary condition
\begin{equation}
g \mathbf{v}_0 = \sum_j m_j \mathbf{v}_j f_j^{(0)} (1 + q_j) d\mathbf{v}_j ,
\end{equation}
where $\mathbf{v}_0$ is the mass average velocity of the gas. Upon insertion of the form for $q_j$, the condition on the $C_j^{ik}$ becomes
\begin{equation}
\sum_j m_j \left( \frac{m_j}{2 k T} \right)^{1/2} \int [ \mathbf{d}_h \| f_j^{(0)} (C_j^{ik} - C_j^{ik}) ] V_j^2 d\mathbf{v}_j
+ \mathbf{d}_h \| f_j^{(0)} (C_j^{ik} - C_j^{ik}) V_j^2 d\mathbf{v}_j
+ (\mathbf{B} \times \mathbf{d}_h \| ) f_j^{(0)} (C_j^{ik} - C_j^{ik}) V_j^2 d\mathbf{v}_j ] = 0 .
\end{equation}

This condition must hold for any orientation of $\mathbf{d}_j$ relative to $\mathbf{B}$, so the sum over $j$ of each term must

\begin{equation}
q_{ij}^{mp} = q_{ij}^{mp} + i \omega_i n_i \left( \frac{2 \pi m_i}{k T} \right)^{1/2} 2 (m + \frac{3}{2}) ! \frac{1}{\pi^{1/2} m !} (m > 0),
\end{equation}
\begin{equation}
\tilde{q}_{ij}^{mm} = \tilde{q}_{ij}^{mm} (m > 0, i \neq j),
\end{equation}
\begin{equation}
\tilde{q}_{ij}^{mp} = q_{ij}^{mp} (m = p, m > 0).
\end{equation}

We see that the magnetic field affects all elements for $m = p = 0$, but occurs otherwise only in diagonal elements. Thus only minor additions to the usual

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\end{equation}
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\end{equation}
\begin{equation}
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\end{equation}
determinant elements are necessary to include the effect of a magnetic field. Expressions for the \( q \) elements in terms of average cross section

\[
\bar{Q}^{(r,s)} = \tau_0^2 \Omega^{(r,s)}
\]

have been given elsewhere \(^{19}\) for \( m, p \leq 3 \). Thus, up to the fourth approximation to the coefficients can be computed.

As already mentioned, the solution for \( A_i^{\perp} \) and \( A_i^{\parallel} \) will not be considered in detail. Suffice it to say that the coefficients \( a_{im} \) in the expansion

\[
A_i = A_i^{\perp} + i A_i^{\parallel} = \sum_{m=0}^{t-1} a_{im} S_{3/2}^{(m)}(W_l^2)
\]

are found from

\[
\sum_j \sum_{p=0}^{t-1} \tilde{g}^{a_{im}}_{j;p} = - \frac{15\tau_{\parallel/}}{2} n_1 \delta_{m1}.
\]

### Transport Coefficients

In view of the similarity of the results of the previous section to those for the usual multicomponent mixture, it is not surprising that the usual expressions for the transport coefficients carry over into this case, with the simple addition of complex terms.

#### Diffusion Velocity

The diffusion velocity of the \( i \)th species is given by

\[
(V_i) = \frac{1}{n_1} \int V_i f_i^{(0)} q_i dV_i
\]

\[
= \frac{n_i^2}{n_1} \sum_j m_j [D_{ij}^{\parallel} d_j^{\parallel} + D_{ij}^{\perp} d_j^{\perp} + D_{ij}^{H}(\hat{B} \times d_j^{\perp})]
\]

\[
- \frac{1}{n_1 m_i T} \left(D_{ij}^{T} \nabla^\parallel T + D_{ij}^{\perp} \nabla^\perp T + D_{ij}^{H} \hat{B} \times \nabla^\perp T\right),
\]

with the perpendicular and Hall components of the diffusion coefficients found from

\[
D_{ij} = D_{ij}^{\perp} + i D_{ij}^{H} = \frac{q n_i}{2 n m_j} \left(2 \frac{k T}{m_i}\right)^{1/2} c_{ij}^{\parallel}
\]

and find from Eq. (31) that

\[
\sum_{j+1} n_j n_j \langle (V_i^*) - (V_i) \rangle = \frac{n^2}{q} \sum_j m_j [n_j D_{ik}^{\parallel} - n_i D_{jk}^{\parallel}] d_k^* = \sum_{j+1} n_j n_j \left(D_{ij}^{T} - D_{ij}^{T}\right) \nabla^* T,
\]

\(38\) See Eqs. (8) and (9) of Ref. \(^{19}\), or the corresponding equations in Ref. \(^{21}\).

\(39\) Ref. \(^{21}\), p. 487.

We see that both complex coefficients are given by the usual multicomponent expressions \(^{38}\) with the replacement of certain of the real determinant elements by the complex elements of Eq. (27). Thus the computation of properties in a magnetic field may be carried out with little more effort than that necessary for the un-ionized multicomponent gas. The task becomes quite simple on computers where complex arithmetic is part of the programming language. The parallel diffusion coefficients are then simply found from the perpendicular components with vanishing magnetic field.

An alternative formulation of Eq. (31) including the binary diffusion coefficients has been obtained by Hirschfelder et al. \(^{39}\). This relation applies here to the parallel diffusion velocities and reads,

\[
\sum_{j+1} n_j n_j \langle (V_i^*) - (V_i) \rangle = \frac{n^2}{2 n^2 T} \sum_{j+1} \frac{n_j n_j}{m_j m_i} \left(D_{ij}^{T} - D_{ij}^{T}\right) \nabla^* T
\]

where the first approximation to the binary diffusion coefficient is given by

\[
D_{ij} = \frac{3}{16 n} \left(\frac{m_i + m_j}{m_i m_j}\right)^{1/2} \left(2 \frac{\pi k T}{m_i}\right)^{1/2} \frac{Q_{ij}^{(1)}}{Q_{ij}^{(1)}}.
\]

If thermal diffusion is neglected, as it may be in many experiments, then Eq. (35) is much simpler than the corresponding form of Eq. (31), since the expressions for \( D_{ij} \) need not be evaluated. A similar expression may also be derived for the gas in a magnetic field. We define a complex diffusion velocity

\[
(V_i^*) = (V_i) - i \hat{B} \times (V_i^{\perp})
\]

and complex driving forces

\[
d_i^* = d_i^{\perp} - i \hat{B} \times d_i^{\perp},
\]

\[
\nabla^* T = \nabla^{\perp} T - i \hat{B} \times \nabla^{\perp} T
\]
where the diffusion coefficients $D_{ij}$ and $D_{ij}^T$ are, of course, complex. In the first approximation ($m = 0$) Eq. (27) can be rewritten in the form

$$
\sum_{j+i} \left[ n_i \sum_{k} \frac{D_{ik}}{m_i} \left( \frac{n_i m_i}{\rho} (\omega_j - \omega_i) + \omega_i \right) \right] \left[ m_k D_{jk}(1) - m_k D_{jk}(1) \right] = \rho (\delta_{ik} - \delta_{jk}), \quad (40)
$$

where the symbol (1) indicates the first approximation is involved. From Eqs. (26) and (32) we obtain

$$
\sum_{j} m_j (m_k D_{jk} - m_k D_{jk}) = 0 \quad (41)
$$

which can be used with $\sum k d_k^* = 0$ and Eq. (40) to obtain

$$
\sum_{j+i} \frac{m_k}{n_i} \left[ n_j D_{ik}(1) - n_i D_{jk}(1) \right] d_k^* = -\frac{q}{kT} \sum_{j+i} \frac{m_i}{n_i} \left( j \times B \right) - e_i n_i \langle V_i \rangle \times B \quad (42)
$$

Equation (42) may now be used in Eq. (39) to eliminate the first term on the right hand side. The resultant expression still contains terms involving $D_{jk}$, which may be eliminated in favour of $j \times B$ and $e_i n_i \langle V_i \rangle \times B$ with Eq. (31). The real part of the final expression reads

$$
\sum_{j+i} \frac{n_i T}{n_i} \left( \langle V_i \rangle^\perp - \langle V_i \rangle \right) = d_i^\perp + \frac{1}{c n_i} \frac{m_i}{q} \left( j \times B \right) - e_i n_i \langle V_i \rangle \times B \quad (43)
$$

where the first two terms involving $D_{ij}^\perp$ and $D_{ij}^{TH}$ are, respectively, the thermal diffusion contributions to $j \times B$ and $\langle V_i \rangle \times B$. It is clear that Eqs. (34) and (43) could be added together to obtain equations for the total diffusion velocities, if desired.

Equation (43), because of its simpler form, is much more readily applicable to many problems than Eq. (31). However, caution must be exercised, since the first approximation to $D_{ij}$ has been used in its derivation. In some cases, the best known being the electron diffusion in a fully ionized gas, this approximation is highly inaccurate. At low ionization in gases where the electron-atom cross section displays the Ramsauer effect, a much higher approximation to the electron diffusion coefficients is needed. Fortunately, it is possible to derive a much simplified form of Eq. (31) and the corresponding transport coefficients for electrons, which can be used to determine the electron diffusion velocity. Equation (43) can still be used for the diffusion velocities of the ions and neutral species. In the case of the diffusion of one type of ion through other ions, errors of about 18% in the ordinary diffusion flux may be introduced by using Eq. (43).

Differences of this size were found between the first and higher approximations to the self-diffusion coefficients of gas composed only of ions. In the case of ions diffusing through atoms, or excited atoms diffusing through groundstate atoms, this equation should be accurate within a few percent.

**Heat Flux**

The heat flux is given by

$$
q \equiv \frac{1}{2} \sum_{j} m_j \int \nu_j V_j^2 \sum \langle \mathbf{V}_j \rangle \ d\mathbf{V}_j \\
= \frac{3}{2} kT \sum_{j} n_j \langle \mathbf{V}_j \rangle - \mathbf{Z}^T \nabla T - \mathbf{Z}^T (\hat{B} \times \nabla T) \\
- n kT \sum_{j} \frac{1}{n_j} \left[ (D_{ij}^T \mathbf{d}_j^\perp + D_{ij}^{TH} \mathbf{d}_j^{\perp}) + (D_{ij}^{TH} \hat{B} \times \mathbf{d}_j^{\perp}) \right] \quad (44)
$$

The derivation of this equation for the parallel components has already given in Ref. 21. The derivation for the other components follows in much the same way. In the reduction of the last terms in Eq. (47) it is convenient to work with the complex quantity $D_{ij}^T \mathbf{d}_j^\perp + D_{ij}^{TH} \mathbf{d}_j^{\perp}$, rather than with each component separately, so that the complex Eq. (15) and its equivalent for $A_i$ may be directly used. It is also
necessary to employ the auxiliary conditions on $C_i^h - C_f^h$ and $A_i$ to effect the reduction.

The parallel coefficient $\lambda^\|_i$ is given by the expressions for the un-ionized gas, while the perpendicular and Hall coefficients are given by

$$\lambda^\perp_i = \lambda^\perp + i \lambda^\text{II} = -\frac{\hat{n}}{\bar{n}} k (2 k T)^{\frac{v_k}{s}} \sum_j \frac{n_j d_{ij}}{m_j^{\frac{v_k}{s}}}$$

and thus can be found from the usual expression upon replacement of the real elements by those of Eq. (28). These coefficients are not the true thermal conductivities, since these are conventionally defined as the coefficient of the thermal gradient when all diffusion fluxes vanish. The various components of $d_i$ must therefore be eliminated in favour of the $d_i^*$ with the aid of Eq. (31). MUCKENFuss and Curtiss accomplishment this for the second approximation, i.e. for two terms in the Sonine polynomial expansion [see Eq. (16)]; it was recently found possible to extend their proof to an arbitrary level of approximation. Through the use of complex arithmetic, these proofs may be easily extended to the gas in a magnetic field. Only the simpler proof for the second approximation will be considered here.

In place of the real vector $\alpha_i$ of Muckenfuss and Curtiss we define the complex vector $\alpha_i^*$ by

$$\alpha_i^* = \frac{2}{n} \left( \frac{m_i}{2 k T} \right)^{\frac{v_k}{s}} \left[ (q_i^*) + \frac{1}{n_i m_i T} D_i^T \nabla^* T \right]$$

where $D_i^T$ and $c_i^h$ are, of course, also complex. We may now solve for $d_i^*$ as done by Muckenfuss and Curtiss for $q^\perp - i \hat{B} \times q^\perp$ which may be formed from Curtiss, and substitute the solution into an expression for $\langle V_i \rangle$. Extracting the real part of this expression, we obtain finally for the heat flux the expression

$$\mathbf{q} = \sum_j \left\{ \frac{5}{2} k T - \frac{\hat{n}}{\bar{n}} k T \sum_i \frac{n_i m_i}{n_i m_i} E_{ij}^\perp \right\} n_j (\langle V_i \rangle)$$

$$+ \left[ \frac{5}{2} k T - \frac{\hat{n}}{\bar{n}} k T \sum_i \frac{n_i m_i}{n_i m_i} E_{ij}^\perp - D_i^T E_{ij}^\text{II} \right] n_j (\langle V_j \rangle)$$

$$- \frac{\hat{n}}{\bar{n}} k T \sum_i \frac{n_i m_i}{n_i m_i} E_{ij}^\perp - D_i^T E_{ij}^\text{II} \right\} n_j (\langle V_j \rangle)$$

$$- \hat{n} \nabla \| T - \hat{n} \nabla T - \hat{n} (\hat{B} \times \nabla T).$$

The complex thermal conductivity $\lambda^\perp + i \lambda^\text{II}$ is given by the expression for $\lambda^\perp_i$ with replacement of determinant real elements by those given in Eq. (28). The matrix $E_{ij}$ is the inverse of the (complex) matrix $D_{ij} m_j$.

$$E_{ij} \equiv E_{ij}^\perp + i E_{ij}^\text{II} = \frac{adj(D_{ij} m_j)}{det(D_{ij} m_j)}.$$

$E_{ij}^\perp$ is the limit of $E_{ij}^\perp$ as $B \to 0$, or may be computed directly from the real diffusion coefficients. Unfortunately, no simple relations corresponding to those for $\lambda^\perp_i$, $\lambda^\perp$ and $\lambda^\text{II}$ have been found for the terms multiplying the diffusion velocities in the general multicomponent mixture. Simple expressions are available for the binary gas and for the contribution of the electrons to this term.

**Comparison with Species Momentum Equations**

In place of the diffusion velocity of each species $\langle V_i \rangle$ SCHRÜTER considers the average velocity

$$\langle v_i \rangle = \frac{1}{n_i} \left( \mathbf{v}_i f_i \frac{d \mathbf{v}_i}{d \mathbf{v}_i} = \langle V_i \rangle - \mathbf{v}_0 \right).$$

An equation for $\langle v_i \rangle$ may be developed from the equation of change of a molecular property,

$$
\begin{align*}
\frac{m_i}{n_i} \frac{D_i \langle v_i \rangle}{D_t} &= -\nabla p_i' - \nabla \cdot \mathbf{\pi}' + n_i \mathbf{X}_i \\
&+ n_i e_i \left( E + \frac{v_i \times B}{c} \right) + R_i
\end{align*}
$$

with

$$p_i' = \frac{\hat{n}}{\bar{n}} n_i m_i \langle (v_i - \langle v_i \rangle)^2 \rangle,$$

$$\mathbf{\pi}'_a = n_i m_i \langle (v_i - \langle v_i \rangle) (v_i - \langle v_i \rangle) \rangle - p_i' \delta_{a0},$$

$$R_i = m_i \sum_j \langle (v_i - \langle v_i \rangle) j_i j_i g \sigma d\omega \mathbf{v}_j \mathbf{v}_i \rangle$$

and

$$\frac{D_i}{D_t} = \nabla + \langle v_i \rangle \cdot \nabla.$$

Before application of Eq. (48) to a specific problem, two assumptions are usually made

$$\mathbf{\pi}_{ia} = 0,$$

$$R_i = -n_i \sum_j n_j e_i \langle (v_i - \langle v_i \rangle) \rangle.$$
In (52a) the stress tensor is neglected in comparison with the static pressure. The form assumed in (52b) for \( R_i \) includes only a friction between the species, with the constants \( \epsilon_{ij} \) to be specified from kinetic theory. A thermal force was also introduced by Maeker and Peters\(^4\) in an attempt to allow for thermal diffusion in the mixture. It can be easily shown from the expressions of the previous sections or from work by Braginskii\(^6,33\) that the form which they use cannot describe thermal diffusion perpendicular to a magnetic field. The thermal diffusion will therefore be neglected in this section. 

With the above assumptions Eq. (48) becomes

\[
\frac{D_i(v_i)}{Dt} = -\nabla p_i + n_i X_i + n_i \epsilon_i \left( \frac{E}{c} + \langle v_i \rangle \times B \right) - n_i \sum_j n_j \epsilon_{ij} \left( \langle v_i \rangle - \langle v_j \rangle \right). \tag{53}
\]

The presence of the velocity difference in Eq. (53) suggests that we compare it with Eqs. (34) and (43). They differ from Eq. (53) in several respects, the most noticeable being the lack of a substantial differential. We can introduce such a substantial derivative with an equation used in the development of the left-hand side of Eq. (1)\(^4\)

\[
\frac{D v_0}{Dt} \equiv \frac{G}{G_t} \left( \frac{2}{2} \right) + v_0 \cdot \nabla v_0 = -\nabla p + \frac{j \times B}{c} + \sum_k n_k X_k. \tag{54}
\]

This equation is, of course, the momentum equation for a gas in the absence of a stress tensor, in the present case with a Maxwellian distribution of velocities. Since the forces causing the diffusion come from the left-hand side of Eq. (1), we may reverse this step and use Eq. (54) to rewrite Eq. (3) for \( d_i \). Combining Eqs. (3), (34), (43) and (54) under the assumption of charge neutrality \( (q_c = 0) \), we obtain

\[
n_i m_i \frac{D v_0}{Dt} = -\nabla p_i + n_i X_i + n_i \epsilon_i \left( \frac{E}{c} + \langle v_i \rangle \times B \right) - \sum_j n_i n_j k T \frac{D_{ij}}{n} \left( \langle v_i \rangle - \langle v_j \rangle \right). \tag{55}
\]

From a comparison of Eq. (53) and (55) we can choose \( \epsilon_{ij} \) as

\[
\epsilon_{ij} = k T / n D_{ij} \tag{56}
\]

In agreement with other workers. However, there are two discrepancies between Eqs. (53) and (55), the most evident being that between the substantial derivatives. They differ by the amount

\[
\frac{2}{2} \langle V_i \rangle - v_0 \cdot \nabla \langle V_i \rangle + \langle V_i \rangle \cdot \nabla (v_0 + \langle V_i \rangle). \tag{57}
\]

When \( v_0 \) vanishes, or is small, and forces causing diffusion are present and possibly changing with time, then it is conceivable that Eq. (57) would not vanish. Fortunately, in many problems treated with Eq. (53), only the steady state is considered, and the other terms vanish identically or are indeed negligible.

The second discrepancy occurs in definition of partial pressure. From the relation between the various velocities we can write

\[
p_i' = p_i - \frac{1}{2} n_i m_i \langle V_i \rangle^2 \tag{48a}
\]

where

\[
p_i = \frac{1}{2} n_i m_i \langle V_i^2 \rangle
\]

is the usual definition of partial pressure\(^1\). The difference can be seen to arise from different definitions of the "therm al" velocity of the gas: in the primed system it is defined relative to the species average velocity \( \langle v_i \rangle \), in the unprimed relative to the mass-average velocity \( v_0 \). If we sum Eq. (48a), we see that the total pressures also differ

\[
p' = p - \frac{1}{2} \sum_i n_i m_i \langle V_i \rangle^2.
\]

Although it is possible to think of situations in which the different definitions could be important (e.g. \( \nabla p_i = 0 \) but \( \nabla p_i' \neq 0 \)), in general they will be negligible. In the first order Chapman-Enskog theory it is assumed that the various gradients and forces are small enough to neglect quadratic terms compared to the linear terms. The diffusion velocities are proportional to the gradients, so terms quadratic in those velocities should indeed be negligible. It may also be noted here that the stress tensors and the temperatures in the two systems also differ by terms proportional to the square of the diffusion velocities.

The derivation of Eq. (55) has also been carried out in a slightly different manner by Johnson\(^26\); however, he did not consider the species momentum equation and thus neglected the important differences brought out here. His derivations, as well as the


\(^{47}\) See Ref. 1, pp. 330—331 for a discussion of this step.
present one, is based on the use of the first approximation, which restricts the application of this equation to species where this level of approximation is accurate. We should not expect to obtain numerically correct factors if we use the above to compute the electron flux in a gas. Braginskii, however, was able to derive an equation similar to (47) for the electron in a fully-ionized gas with inclusion of thermal diffusion forces in $R_j$. The above derivation of Eq. (55) from kinetic theory allows us to set quite generally its limits of validity, namely, those of the Chapman-Enskog method: mean-free-path and time between collisions shorter than distances and times characteristic of the fluid gradients, and those mentioned earlier with regard to the collision integral and thermal diffusion. Such general statements are not possible in the derivation from a species momentum equation. The species momentum equation does have the advantage of being able to describe wave phenomenon in which electron or ion inertia is important. Such cannot be accomplished with the usual Chapman-Enskog method.

It is evident from Eqs. (34) and (43) that we could have included terms representing thermal diffusion in Eq. (55). Such a step does not seem advantageous for several reasons. Firstly such terms in these equations perpendicular to the magnetic field are quite complicated, and would be difficult to use in practical applications. Secondly, since there is no reasonably simple relation between the thermal diffusion and binary diffusion coefficients, the full multicomponent expressions described earlier must be used to compute $D_j^T$. But then much of the simplicity of the method of this section is lost and one would be better-advised to use instead Eq. (31) for the diffusion velocity. This becomes more evident when we consider that $D_j^T$ will almost always be computed with a higher approximation than $D_j$, so little additional effort is required to compute the multicomponent diffusion coefficients simultaneously with the computation of those for thermal diffusion.

**Discussion**

We have seen how it is possible to modify the usual expressions for transport coefficients of multicomponent gases to allow for ionization in the presence of a magnetic field. The modifications involve only the introduction of complex elements containing the cyclotron frequencies into the determinants. Nonetheless, the practical application of these expressions to a plasma containing electrons becomes a complicated task since, in order to compute accurate coefficients for the mixture, at least the second approximation must be used for the electrical conductivity and the third approximation for the thermal conductivity of ionized gases. This slow convergence, relative to un-ionized gases, is caused by the presence of the electrons in the mixture. It is possible to separate the contribution of the electrons to the transport coefficients from that of the ions and neutrals, and so to use a lower approximation for the latter than for the electrons. The present expressions are then valuable for computing the properties of the mixture of ions and neutrals, and as a starting point for deducing simplified expressions for the electron properties. These simplified expressions will be developed in a paper to follow.

I am grateful to Prof. R. Wienecke for his encouragement of this research, and to the National Science Foundation for the award of a postdoctoral fellowship, during the period of which most of this work was carried out. — This work has been undertaken as part of the joint research program of the Institut für Plasmafysik and Euratom.