Scattering of Electromagnetic Waves by Turbulent, Weakly Ionized Plasmas

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The scattering of a plane, monochromatic electromagnetic wave by the fluctuations of the dielectric constant of a turbulent weakly, ionized plasma is investigated on the basis of the statistical theory of locally isotropic turbulence. The effects of collisions on the fluctuating dielectric constant of the electrons are taken into account. The spectral density determining the scattering cross section is related to the structure functions of the fluctuations of the electron concentration and the temperature. These structure functions are calculated from the transport equations with the use of methods developed in the statistical theory of strong turbulence.

In this contribution we will use the statistical theory of locally isotropic turbulence. The effects of collisions on the fluctuating dielectric constant are investigated on the basis of the statistical theory of strong turbulence. These investigations consider the turbulent plasma as an incompressible fluid and calculate the scattering due to fluctuations of the electron density. They do not account for the effects of particle collisions on the fluctuating dielectric constant.

In this contribution we will use the statistical theory of locally isotropic turbulence developed by Kolmogorov, Obukhov and Yaglom to determine the scattering cross section of a weakly ionized, collision dominated plasma with temperature and density fluctuations. We consider strong turbulence in the sense of classical hydrodynamics.

System

Subject of this investigation is the scattering of a plane, monochromatic electromagnetic wave

\[
E = E_0 e^{i(\omega t - k r)}
\]

by the randomly fluctuating scalar dielectric constant of a turbulent plasma within a volume \( V \). The plasma is weakly ionized and collision dominated (seeded atmosphere). We assume that it can be described within the frame of the first order of the Chapman-Enskog development (hydrodynamic approximation). We neglect heat transfer by radiation and by electron diffusion as well as dissipation of mechanical energy by viscous forces. The turbulence is considered to be homogeneous and locally isotropic.

Concept

With the use of Maxwell's equations the scattering cross section is expressed by the spectral density of the fluctuating dielectric constant. Accounting for the effects of electron-neutral and electron-ion collisions we relate the fluctuations of the electron dielectric constant to the fluctuations of the temperature and the electron concentration. The spectral densities of these fluctuations are calculated from the transport equations using the statistical theory of locally isotropic turbulence.

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Analysis

Scattering Cross Section

The cross section $\sigma d\Omega$ for the scattering of a plane, monochromatic wave per unit plasma volume into the solid angle $d\Omega$ around the spatial direction $k'$ is defined by

$$\sigma d\Omega = \frac{R^2}{V} |E'|^2 / |E_0|^2 d\Omega.$$ \hspace{1cm} (2)

$R$ is the distance from the scattering volume to the observation point, which is supposed to be large in comparison to the dimensions of the scattering region ($R \gg V^{1/3}$). $E'$ is the electric field in the wave scattered in the direction $k'$. The prime indicates quantities referring to the scattered wave.

We assume that $E'$ and the electric induction $D'$ are small in comparison to the corresponding fields in the incident wave and that the fluctuating part $\varepsilon_1$ of the dielectric constant is small in comparison to its average value $\bar{\varepsilon}$.

$E'$ can then be calculated from Maxwell's equations via Fourier analysis in time and space \(^4\).

$$E' = \frac{-\exp i(\omega t - kR)}{R} \cdot k' \times k' \times \int \varepsilon_1 E_0 \exp(iK r) \, dr.$$ \hspace{1cm} (3)

Inserting this expression into equation (2) we find

$$\sigma d\Omega = k^4 \sin^2 \chi \cdot \left( \int \varepsilon_1(r_1) \varepsilon_1(r_2) e^{iK(r_1 - r_2)} \, dr_1 \, dr_2 \right) \, d\Omega.$$ \hspace{1cm} (4)

$\varepsilon_1(r_1) \varepsilon_1(r_2)$ is the correlation function $C_\varepsilon$ of the refractive index fluctuations. Since our $\varepsilon$-field is homogeneous $C_\varepsilon$ depends only on $r_1 - r_2$. Introducing the spectral density

$$S_\varepsilon(K) = \int C_\varepsilon(r_1 - r_2) e^{iK(r_1 - r_2)} \, dr_1 \, dr_2$$ \hspace{1cm} (5)

we finally obtain

$$\sigma d\Omega = k^4 \sin^2 \chi S_\varepsilon(K) \, d\Omega$$ \hspace{1cm} (6)

and the evaluation of the scattering cross section is reduced to the problem of determining the spectral density of the fluctuations of the dielectric constant.

Dielectric Constant

We consider here the scattering due to fluctuations of the dielectric constant $\varepsilon^{(e)}$ of the free electrons. The contribution of the ions to $\varepsilon$ may be neglected due to the large ion-electron mass ratio. The contribution of the neutrals becomes important only if the degree of ionization is extremely small.

The dielectric constant $\varepsilon^{(e)}$ is conveniently written in the form \(^5\)

$$\varepsilon^{(e)} = 1 - g\left(\frac{\omega}{\tilde{\nu}}\right) \frac{4\pi e^2 n_0}{m} \frac{1}{\omega^2 + \tilde{\nu}^2}.$$ \hspace{1cm} (8)

where $g(\omega/\tilde{\nu})$ is a slightly varying function with values close to 1 and the effective collision frequency $\tilde{\nu}$ is defined by

$$\tilde{\nu} = \bar{\nu}_0 + \bar{\nu}_i = \frac{2}{3\sqrt{2\pi}} \left(\frac{m}{KT}\right)^{5/2} \int_0^\infty \left[\nu_0(v) + \nu_i(v)\right] v^4 e^{-mv^2/2KT} \, dv.$$ \hspace{1cm} (9)

$\bar{\nu}_0$ and $\bar{\nu}_i$ designate the effective frequencies of electron-neutral and electron-ion collisions resp. Neglecting the influence of the Ramsauer effect on $\nu_0$ and evaluating $\nu_i$ with Rutherford's formula for the electron-ion collision cross section one obtains \(^5\)

$$\bar{\nu}_0 = 8.3 \cdot 10^5 \pi e^2 \sqrt{T \, n_0}, \quad \bar{\nu}_i = \frac{5.5}{T^{3/2} n} \ln \left(\frac{220 T}{n^{1/3}}\right).$$ \hspace{1cm} (10)


Where $\pi \rho^2$ is an effective cross section. We now combine equs. (8) and (10). Introducing the electron concentration $c = n_e / n$ and the ideal gas law we find with the assumptions of quasi-neutrality and weak ionization

$$e^{(e)} = 1 - \frac{4 \pi e^2}{m_K T} \frac{\omega^2}{\omega^2 + \left(8.3 \cdot 10^5 \pi q^2 / 5.5 e n_{1/3} K_T^{1/2} + \frac{220 T}{n_{1/3}} \right)}.$$  (11)

**Transport Equations**

Using the assumptions that the turbulence is homogeneous and that the fluctuations in electron concentration, temperature and pressure are small and not correlated with each other we find from equ. (11)

$$S_e = \left(\frac{\partial e}{\partial c}ight)_y S_c + \left(\frac{\partial e}{\partial T}ight)_y S_T + \left(\frac{\partial e}{\partial p}ight)_y S_p, \quad y = \{c; T; p\}. \quad (12)$$

To determine the spectral densities of $c$ and $T$ we start from the corresponding transport equations. By taking appropriate moments of the Boltzmann equation we find up to first order in the Enskog development the following balances for particle densities, total momentum and total internal energy

$$\frac{\partial n_x}{\partial t} + \nabla \cdot (n_x \mathbf{v}) + \nabla \cdot (n_x U_x) = R_x, \quad \varrho \frac{d \mathbf{v}_i}{dt} = - \sum_j \frac{\partial}{\partial x_j} p_{ij}, \quad (13, 14)$$

$$\frac{3}{2} \varrho \frac{d}{dt} \left(\frac{n x K T}{\varrho}\right) = \sum_{ij} p_{ij} \frac{\partial v_i}{\partial x_j} + \nabla \cdot (\varrho \chi \text{grad} T) - \frac{5}{2} \nabla \cdot \left(\sum_{z} n_z U_z K T\right)$$

With

$$n_x U_x = - n \chi \text{grad} \ln T - m_\beta \frac{n_x}{\varrho} n^2 D \text{grad} c_x + \frac{m_\beta n^2 D \varrho}{\varrho} \left(1 - \frac{n m_\beta}{\varrho}\right) \text{grad} \ln p$$  (16)

and

$$p_{ij} = \varrho \chi \frac{\partial}{\partial t} - \mu \left(\frac{\partial v_i}{\partial x_j} + \frac{\partial v_j}{\partial x_i}\right) - \left(\mu_\beta - \frac{2}{3} \mu\right) \delta_{ij} \nabla \cdot \mathbf{v}, \quad p = n K T. \quad (17, 18)$$

The indices $\alpha$ and $\beta$ denote the different particle components. We distinguish here only two components: electrons and neutrals. This is possible, since we assume that the plasma is weakly ionized and we are not interested in specific effects of the ions. The ions may be treated on equal grounds with the neutrals. $n$ is the total particle density and $c_x = n_x / n$ the particle concentration of component $x$. $\varrho$ is the mass density of the plasma and $\mathbf{v}$ the center of mass velocity. $R_x$ denotes the particle production rate, $\chi$ the heat conductivity. $D, \chi, \mu, \mu_\beta$ are the coefficients of diffusion, thermal diffusion, viscosity and bulk viscosity resp.

**Fluctuations of the Electron Concentration**

Using our assumptions of weak ionization and small variations in temperature and density, equs. (13)—(18) may be considerably simplified. For the electron component the last term of equ. (16) vanishes due to weak ionization. Since the thermal diffusion coefficient $\chi$ is proportional to the electron concentration $c_-$ the first term on the right-hand side of equ. (16) is small of second order and may therefore also be neglected. Consequently the particle balance for the electrons simplifies to

$$\frac{\partial n_-}{\partial t} + \nabla \cdot (n_- \mathbf{v}) - \nabla \cdot (n D \text{grad} c_-) = R_-.$$  (19)

Multiplying equ. (19) with $m_-$ and using the continuity equation

$$\frac{\partial \varrho}{\partial t} + \nabla (\varrho \mathbf{v}) = 0$$  (20)

we find for the mass concentration $\gamma_-$ of the electrons

$$\frac{\partial \gamma_-}{\partial t} + \varrho \mathbf{v} \text{grad} \gamma_- - m_- n D \Delta c_- = m_- R_-.$$  (21)

In writing down the diffusion term of equ. (21) we have again made use of our assumption that the relative gradients of density and temperature are small. With the relations

$$m_- c_- / m_0 \approx \gamma_-, \quad \varrho \approx m_0 n$$

which hold due to weak ionization, we finally have
\[ \frac{\partial c_-}{\partial t} + \mathbf{v} \text{grad} c_- - D \Delta c_- = \frac{R-i}{n}. \]  
(23)
Starting from equ. (23) we can use a procedure developed by Yaglom\(^7\) to derive the structure function of the fluctuations of the electron-concentration. To do this we have to assume that the velocity field is approximately solenoidal, so that we can use the relation
\[ \nabla \cdot \mathbf{v} = 0. \]  
(24)
Then we multiply equ. (23) with \( c'_-(r_2) \) and add the corresponding equation with the primed and unprimed coordinates interchanged. We obtain
\[ \frac{\partial c'_-}{\partial t} = -2 \sum \frac{\partial}{\partial \xi_j} v_j c'_- + 2 D A c'_- + 2 \alpha c c'. \]  
(25)
where \( \alpha = \) is the number of ionizing collisions per electron and unit time, \( \xi_j \) are the components of the vector \( r = r_1 - r_2 \) and the Laplacian \( \Delta \) has to be taken with respect to these components. We now introduce the correlation functions \( C_{cc}, C_{1c2}, B_{cc} \) and the structure functions \( B_{ccc}, B_{1cc} \) defined by
\[ C_{cc} = \langle c_r(r_1) c_r(r_2) \rangle, \quad C_{1c2} = \langle v_t(r_1) c_2(r_2) \rangle, \]  
(26)
\[ B_{cc} = \langle (c_r(r_1) - c(r_2))^2 \rangle, \quad B_{1cc} = \langle (v_t - v_t')(c-c')^2 \rangle, \]  
where \( v_t \) is the velocity component in the direction of \( r \). Due to assumption (24) \( C_{1c2} \) vanishes\(^8\).
\[ C_{1c2} = 0. \]  
(27)
We further have
\[ \frac{\partial C_{cc}(r)}{\partial t} = \frac{\partial C_{cc}(0)}{\partial t}. \]  
(28)
This relation follows from the equality
\[ B_{cc}(r) = 2(C_{cc}(0) - C_{cc}(r)) \]  
(29)
for a homogeneous and isotropic random field and the assumption that the structure function is time-independent.

From equ. (25) we obtain with the use of (27), (28) and (29)
\[ \frac{1}{r^2} \frac{d}{dr} r^2 B_{1cc}(r) = \frac{2D}{r^2} \frac{d}{dr} r^2 \frac{dB_{cc}(r)}{dr} - 2\alpha B_{cc}(r) \]  
\[ = 2\alpha \frac{\partial}{\partial t} C_{cc}(0) - 4\alpha C_{cc}(0). \]  
(30)
This hierarchy equation is closed by the assumption of constant asymmetry \( \alpha \).
\[ B_{1cc} = -\alpha B_{cc} \sqrt{B_{ll}}. \]  
(31)
\( B_{cc} \) can then be calculated provided that the longitudinal structure function \( B_{ll} \) of the velocity field is known.
\[ B_{ll} \text{ is determined by the Kolmogorov equ.}\(^9\). \]  
\[ \frac{dB_{ll}}{dr} \bigg|_{r=0} = 0 \]
(32)
which follows from the Navier-Stokes equ. (14). With the assumption of constant asymmetry
\[ B_{ll} = -\alpha_l (B_{ll})^{3/2} \]  
(33)
one obtains from equ. (32) the relations
\[ B_{ll} = \alpha_1 r^2 = \frac{1}{10} \left( \frac{\partial C_{ll}(0)}{\partial t} \right) r^2, \quad r < r_0 = \left( \frac{\beta_l}{\alpha_l} \right)^{3/4}. \]  
(34)
\[ B_{ll} = \beta_l r^{2/3} = \left( \frac{6}{5} \alpha_l \right)^{2/3} r^{2/3}, \quad r > r_0. \]  
(35)
Limiting \( r \) to small values such that equ. (34) holds and the conditions
\[ 4D/r^2 > a_c/\sqrt{\alpha_1}, \quad O(a_c/\sqrt{\alpha_1}) = O(\alpha) \]  
are satisfied, equ. (30) may be simplified to
\[ \frac{d^2 B}{dr^2} + \frac{1}{r} \frac{d B}{dr} + \left( \frac{3}{2} a_c \sqrt{\alpha_1} + \alpha \right) B = -\frac{A}{2D} \]  
(37)
where the constant \( A \) is given by
\[ A = -2 \frac{\partial C_{cc}(0)}{\partial t} - 4\alpha C_{cc}(0). \]  
(38)
With the boundary condition
\[ (d B/dr)_{r=0} = 0 \]  
(39)
\(^7\) A. M. Yaglom, Dokl. Akad. Nauk SSSR 69, 743 [1949].
we find the solution
\[ B_{cc}(r) = - \frac{A}{12D} r^2, \quad r < \text{Min} \left\{ r_0, \left( \frac{4D}{a_c \sqrt{x_1}} \right)^{1/2} \right\}. \] (40)

In the range of large \( r \) values we combine equ. (35) with equ. (30) and transform to the new variable
\[ x = r^{2/3}. \]
We then obtain
\[ \frac{d^2 B(x)}{dx^2} - \left( \frac{5}{2x} + \frac{3}{4} \frac{a_c \sqrt{\beta_1}}{D} x \right) \frac{dB(x)}{dx} + \left( \frac{21}{8} \frac{a_c \sqrt{\beta_1}}{D} + \frac{9x}{4D} \right) B(x) = - \frac{9}{8} \frac{A}{D} x. \] (41)

If the conditions
\[ D|x^2 \ll \frac{1}{2} a_c \sqrt{\beta_1}, \quad a_c \sqrt{\beta_1}|x| \gg x \] (42)
are satisfied the solution of equ. (41) is
\[ B_{cc}(x) = \exp \left\{ -\frac{3}{16} \frac{a_c \sqrt{\beta_1}}{D} x^2 \right\} \left[ C_1 M_{3/2, 1/4} \left( \frac{3}{8} \frac{a_c \sqrt{\beta_1}}{D} x^2 \right) + C_2 M_{3/2, -1/4} \left( \frac{3}{8} \frac{a_c \sqrt{\beta_1}}{D} x^2 \right) - \frac{1}{3} \frac{A}{a_c \sqrt{\beta_1}} x \right], \] (43)
where \( M \) designate Whittaker's functions. The constants \( C_1 \) and \( C_2 \) are to be determined by continuity requirements at the lower limit of the validity range of (43).

Due to the inequalities (42) and the asymptotic expansion
\[ M_{3/2, \nu}(Z) = \frac{\Gamma(2\nu + 1)}{\Gamma(\nu - 1)} e^{Z/2} Z^{-3/2} \{ 1 + O(Z^{-1}) \}, \quad Z \to \infty \] (44)
the solution (43) is reduced to
\[ B_{cc}(r) = -\frac{1}{3} \frac{A}{a_c \sqrt{\beta_1}} r^{2/3} = \beta_c r^{2/3}, \quad r > r_{0c} = \text{Max} \left\{ r_0; \left( \frac{4D}{a_c \sqrt{x_1}} \right)^{3/4} \right\}, \] (45)

For \( x \) values satisfying the conditions
\[ D|x^2 \ll \frac{1}{2} a_c \sqrt{\beta_1}, \quad a_c \sqrt{\beta_1}|x| \ll x, \] (46)
the solution of equ. (41) is given by
\[ B_{cc}(x) = \exp \left\{ -\frac{3}{16} \frac{a_c \sqrt{\beta_1}}{D} x^2 \right\} \eta(\xi) - \frac{A}{2x} \] (47)
where \( \xi \) is defined by
\[ \xi = \sqrt{\frac{3}{4} \frac{a_c \sqrt{\beta_1}}{D} \left( x - \frac{8\pi}{a_c \sqrt{\beta_1}} \right)} \] (48)
and \( \eta \) is the solution of the equation
\[ \eta'' + \xi \eta + \frac{12\pi^2 D}{(a_c \sqrt{\beta_1})^3} \eta = 0. \] (49)

Equ. (49) may be reduced to Whittaker's equation. Inserting the solution into (47) we see that in the range limited by (46) \( B_{cc} \) may be approximated by
\[ B_{cc} = -A/2x, \quad r > r_{1c} > r_{0c}. \] (50)

As is easily verified the solutions (45) and (50) are asymptotic expansions of the function
\[ B_{cc}(r) = - \frac{A}{2x} \left\{ 1 - \frac{2^{2/3}}{\Gamma(1/3)} \left( \frac{r}{r_1} \right)^{1/3} K_{1/3} \left( \frac{r}{r_1} \right) \right\} \] (51)
where \( K \) is the modified Bessel function of the second kind and the normalizing length \( r_1 \) is defined by
\[ r_1 = \left( \frac{3}{2} \right)^{3/2} \cdot r_{1c} = \left( \frac{3}{2} \frac{a_c \sqrt{\beta_1}}{x} \right)^{3/2}. \] (52)

We will therefore use equ. (51) as structure function of the turbulent fluctuations of the electron concentration in the whole range \( r > r_{0c} \). The correlation is related to the structure function by equ. (29).

Temperature Fluctuations

We will now calculate the structure function of the temperature fluctuations. Neglecting the heat transfer due to particle diffusion and the heat production due to dissipation of mechanical energy by viscous forces we find from equ. (15) with the use of equus. (13) and (24)
\[ \frac{dT}{dt} = -\frac{2}{3} T \nabla \cdot v + \frac{1}{K} \frac{\nabla (s \nabla T)}{3n} \frac{\nabla T}{n} \sum R_x. \] (53)

Within the frame of our model equ. (53) may be further simplified to
\[ \frac{dT}{dt} + v \nabla T - \frac{2}{3} \frac{\nabla T}{n} \Delta T = 0. \] (54)

This equation has the same structure as equ. (23) without the production term. Therefore we use the same procedure as in the derivation of equ. (30). With the boundary conditions
\[ B_{TT}(0) = \frac{d}{dr} B_{TT} |_{r=0} = 0 \] (55)
we obtain
\[ B_{TT} = - \frac{4}{3} \int \frac{d B_{TT}}{d r} = - \frac{2}{3} r \frac{\partial C_{TT}(0)}{\partial t} \quad (56) \]

This equ. too is closed by assuming constant asymmetry
\[ B_{TT} = - a_T B_{TT} \sqrt{B_{II}} \quad (57) \]

Inserting the longitudinal structure functions (34) resp. (35) we find
\[ B_{TT}(r) = a_T r^2 = \left( \frac{n}{4} \frac{x'}{n} \frac{\partial C_{TT}(0)}{\partial t} \right) r^2, \quad r < \text{Min}\left\{ r_{01} ; \left( \frac{8}{3} \frac{x'}{n} \frac{1}{\sqrt{\alpha_T}} \right)^{1/2} \right\}, \quad (58) \]
\[ B_{TT}(r) = \beta_T r^{2/3} = \left( \frac{2}{3} \frac{\partial C_{TT}(0)}{\partial t} \right) r^{2/3}, \quad r > r_{0T} = \text{Max}\left\{ r_{01} ; \left( \frac{8}{3} \frac{x'}{n} \frac{1}{\sqrt{\alpha_T}} \right)^{3/4} \right\}, \quad (59) \]

**Spectral Densities**

Knowing the structure functions \( B_{cc} \) and \( B_{TT} \) we can now determine the spectral densities via the relation
\[ B_{yy} = 2 \int_{-\infty}^{\infty} \left( 1 - \cos(Kr) \right) S(K) dK. \quad (60) \]

\[ S_e = \frac{\Gamma(8/3)}{4 \pi^2} \left( \frac{\pi}{3} \right) \left( \frac{\varepsilon - 1}{c} \right)^2 \left( \frac{2 \nu \bar{v}_i}{\omega^2 + \nu^2} - 1 \right)^2 A \frac{r_i^3}{2 \pi \left( 1 + K^2 r_i^2 \right)^{11/6}} \]
\[ S_T(K) = \frac{\Gamma(8/3)}{4 \pi^2} \left( \frac{\pi}{3} \right) \beta_T K^{-11/3} \quad (61) \]
\[ \Delta p = - \sum_{i,j=1}^{3} \frac{\varepsilon_{x_i} \varepsilon_{x_j}}{\varepsilon_{x_i}} \quad (63) \]

But the effect of pressure fluctuations on the refractive index fluctuations is negligible in comparison to temperature and electron density fluctuations. Thus we have from equ. (12)

\[ R_0 = \text{Max}\left\{ r_{0c} ; r_{0T} \right\} , \quad K < 1/R_0 \]

where all fluctuating quantities are to be represented by their average values. The lower \( K \) limit for the applicability range of (64) is determined by the validity of the assumptions of homogeneity and local isotropy.

The final result for the scattering cross section \( \sigma \) is obtained by inserting equ. (64) into equ. (7). Introducing the scattering angle via
\[ K = 2 k \sin \frac{\theta}{2} \quad (65) \]
we have for \( 1/R_0 > K > 1/r_i \)
\[ \sigma d\Omega = 2.6 \cdot 10^{-3} \left( \frac{\varepsilon - 1}{c} \right)^2 \left( \frac{2 \nu \bar{v}_i}{\omega^2 + \nu^2} - 1 \right)^2 \beta_e \]
\[ + \left( \frac{\varepsilon - 1}{T} \right)^2 \left( \frac{v \nu_0 + 3 \nu_i}{\omega^2 + \nu^2} - 1 \right)^2 \beta_T \left( \sin \frac{\theta}{2} \right)^{-11/3} \sin^2 \chi d\Omega. \quad (66) \]

**Discussion**

On the basis of the statistical theory of locally isotropic turbulence we have rigorously calculated the scattering cross section of a weakly ionized, collision dominated turbulent plasma in the hydrodynamic regime. The hypothesis of constant asymmetry, which closes equs. (30), (32) and (56), has been supported by experimental findings. The absolute values \( a_i, a_e, a_T \) of the asymmetries enter-

ing the characteristics $\beta_c$, $\beta_T$ of the fluctuating quantities $c$ and $T$ in equ. (66) remain undetermined within the frame of the theory used in this investigation. They have to be fixed either by experiments or by additional theoretical assumptions. The other quantities $\partial c^2/\partial t$, $\partial T^2/\partial t$ on which $\beta_c$ and $\beta_T$ depend, can be related to the mean square gradients\footnote{A. M. OBUKHOV, Izv. Akad. Nauk SSSR, Ser. Geograf. Geofiz. 13, 58 [1949]. German translation in "Sammelband zur statistischen Theorie der Turbulenz", Akademie-Verlag, Berlin 1958.}

$$\begin{align*}
\frac{\partial}{\partial t} T^2 &= -\frac{4}{3} \pi' \left(\frac{\text{grad} T}{}\right)^2, \\
\frac{\partial}{\partial t} c^2 &= -2 D (\text{grad} c)^2.
\end{align*}$$

(67)

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Dynamische Theorie der Röntgenstrahl-Interferenzen an schwach verzerrten Kristallgittern
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\begin{flushright}
II. Strahlenoptik von Bloch-Wellen im allgemeinen Fall und im Zweistrahlfall
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A geometrical-optical approach to the problem of the propagation of X-rays in weakly deformed crystals is developed in a general form along the lines of the Hamilton-Jakobi theory. It starts from the eikonal equation, which has been derived from Maxwell's equations in Part I\footnote{K. KAMBE, Z. Naturforsch. 29a, 770 [1965].} and arrives at the equations of PENNING and POLDER\footnote{M. BORN u. E. WOLF, Principles of Optics, Pergamon Press, London 1959. Kap. 3.} and Fermat's principle by KATO\footnote{N. KATO, J. Phys. Soc. Japan 18, 1785 [1963]; 19, 67, 971 [1964]. Das Fermatische Prinzip wurde von KATO aus den Maxwell'schen Gleichungen mit Hilfe von „modifizierten Blochschen Funktionen“ ebenfalls über eine Eikonalgleichung hergeleitet.}. The amplitude equation, which has also been derived in Part I, is interpreted by means of the energy-flow picture, the absorption being taken into account.

Application of the theory to the two-beam case gives an equation of rays, which has a remarkable resemblance to the relativistic equation of motion of charged particles in an electromagnetic field. Representation of the lattice distortion by a displacement vector enables one to obtain equations of rays, phases and amplitudes in a form suited to the practical calculation.

Im ersten Teil\footnote{Abteilung Prof. Dr. K. Molière.} dieser Arbeit wurde gezeigt, daß sich die Fortpflanzung von Röntgen-Strahlen in schwach verzerrten Kristallgittern durch eine strahlenoptische Nähe rung beschreiben läßt. Durch den gleichzeitigen Übergang zu sehr hohen Frequenzen und sehr kleinen Gitterperioden wurden aus der Wellengleichung eine Eikonalgleichung und die zugehörige Amplitudengleichung abgeleitet.


Nachdem die Theorie im Teil A der vorliegenden Arbeit in allgemeiner Form entwickelt worden ist, wird sie im Teil B auf den Zweistrahlfall, nämlich den Interferenzfall an einer Netzebene, angewandt.