1. Introduction

Quantum mechanics has been founded as a refinement of classical mechanics mainly for low energies (small particle numbers). In the limit of high energies, quantum formulae have to tend towards their classical counterparts. Especially, Boltzmann statistics has to tend towards Gibbs statistics for particle energies large compared with some characteristic energy $kT_0$. Which can be stated, e.g. as an asymptotic equality of the partition sums $Z(T)$ of the respective statistics.

$$Z_G(\beta) := \int dm \ e^{\beta H(p,q)}, \ \beta = -1/kT, \ (1)$$

where the indices G and B stand for “Gibbs” and “Boltzmann”, $H(p,q)$ is Hamilton’s energy function on 2 f-dim phase space with volume element $dm := h^{-1} \prod_j dp_j \prod_k dq_k$, $(h = \text{Planck’s constant}), \ (3)$

and where $H_j$ stands for the eigenvalues of the corresponding Hamilton operator, and $w_j$ for the dimension of the eigen space belonging to $H_j$ (called its statistical weight).

The partition integral $Z_G$ can also be written as a one-dim energy integral

$$Z_G(\beta) = \int dH \ m'(H) \ e^{\beta H}, \ (4)$$

which is approximated by the sum (2) if $w_j e^{\beta H_j}$ approximates the integrand $m'(H) \ e^{\beta H}$ up to a constant factor. The latter is similar in shape to $H \ e^{-\beta(H)}$ whose essential support is a “thin” neighbourhood of the energy surface $H = f/|\beta| = f kT$. Consequently, $Z_B$ approaches $Z_G$ for large energies if $w_j = w(H_j)$ approaches $dm(H_j)$ for $H_j \gg f kT_0$.

In words: quantum statistics approaches classical statistics for large energies if the eigenvalue density, or statistical weight of the energy shell approaches its volume (measured in units of $h$).

Such an asymptotic equality is inherent in Born’s heuristic quantization rule

$$\langle \psi | H(p,q) | \psi \rangle = \frac{\hbar}{2\pi} \delta(p - \hat{p}) \Rightarrow 0 \ (5)$$

which implies for one-dim systems that neighbouring “quantum orbits” divide phase space into cells of constant volume $h$. It is the purpose of the present paper to investigate the scope of its validity for arbitrary Hamilton functions quantized according to the Weyl–Wigner correspondence.

The correspondence of Weyl and Wigner gives a quantization rule $\Phi$ which is invariant under continuous coordinate changes in position space. That is

$$\Phi : H(p,q) \rightarrow \mathbf{H} \ (6)$$

is a mapping of real phase space functions $H(p,q)$ onto selfadjoint Hilbert space operators $\mathbf{H}$ which defines a Hamilton operator $\mathbf{H}$ for every Hamilton function $H$ such that continuously related position space descriptions give rise to unitarily related quantum theories; proofs are given in 1. This rule is of course not unique (because of the freedom to perform unitary transformations), but it is the simplest rule known to me. Its properties will be described in section 2, and used for a comparison of the statistical weight with the energy shell volume in section 3.

1 W. KUNDT, Springer Tracts in Modern Physics 40, 107 [1966].
2. Canonical Quantization

In this section we summarize the relevant properties of the invariant quantization rule \( \Phi \) suggested by Weyl, and used by Wigner (and many others) for a phase space description of a quantized system (proofs and references are given in \(^1\)).

\( \Phi \) is conveniently introduced as a mapping of square integrable (complex) phase space functions \( H(p, q) \) onto Hilbert space \( [\text{tr}(H^*H) < \infty] \), which is based upon Fourier transformations. As such it is a linear isomorphism which maps the phase space scalar product

\[
\langle A, B \rangle := \int \mathrm{d}m \, A^\dagger(p, q) \, B(p, q)
\]

onto the operator scalar product

\[
\langle A, B \rangle := \text{tr}(A^\dagger \, B), \quad (A^\dagger := \text{adjoint}(A)).
\]

In other words, \( \Phi \) is a Hilbert space isomorphism. Moreover, its inverse \( \Phi^{-1} \) maps the commutator of operators

\[
[A, B] := (i/\hbar) \,(A \, B - B \, A)
\]

onto the Moyal bracket of phase space functions:

\[
[A, B] := (2/\hbar) \sin\left(\frac{1}{2} \, \hbar \, \Gamma\right) \, A \, B.
\]

Here, \( \Gamma \) is the Poisson operator which maps two functions \( A, B \) onto their Poisson bracket

\[
[A, B]_\Gamma := \Gamma \, A \, B := \sum_i (\partial_{p_j} \, B \partial_{q_j} - \partial_{q_j} \, B \partial_{p_j}) \, A \, B,
\]

where an upper kernel index \( A \) says that this differentiation operator acts on the function \( A \) only. Like the commutator, the Moyal bracket (10) is a Lie product. It is non-local because it is formed from the second order differential operator \( \Gamma \) by insertion into the (infinite) power series expansion of the sine-function. However, the Moyal bracket differs from the Poisson bracket by terms of second degree in \( \hbar \) only, and is identical with it if \( A \) or \( B \) are polynomials of maximal degree two in \( p_j \) and \( q_k \). Consequently, \( \Phi \) is also a Lie algebra isomorphism (whenever the Lie product is defined) if one uses the Moyal bracket as a Lie product on the phase space functions.

In what follows, we do not need the definition of \( \Phi \) (which is lengthy). Instead we observe that \( \Phi \) can be extended to arbitrary analytic functions, and that this unique extension is given by (\( [P_j, Q_k] = \delta_{jk} \) for the commutator defined in (9))

\[
\Phi \, A = \exp\left(-\frac{i}{\hbar} \sum_j \partial_{p_j} \partial_{Q_j}\right) A_L(P, Q); \quad (12)
\]

here the left ordered operator \( A_L(P, Q) \) is obtained from the power series \( A(p, q) \) by replacing all \( p_j, q_k \) by their corresponding operators \( P_j, Q_k \) such that all the \( Q_j \)'s stand to the left of the \( P_k \)'s. As an immediate consequence one sees that functions of the \( p_j \)'s or \( q_k \)'s alone are mapped under \( \Phi \) onto the same functions of the corresponding operators:

\[
\Phi \, A(p) = A \,(P), \quad \Phi \, A(q) = A \,(Q). \quad (13)
\]

We remark that the effect of the map (12) is to replace polynomials by their corresponding totally symmetrized operator polynomials.

For later use we mention the following example calculated from (12)

\[
\Phi(p \, q^2)^2 = Q^2 \, P^2 - 4 \, i \, \hbar \, Q^3 \, P - 3 \, \hbar^2 Q^2 = \frac{1}{4} \,(Q^2 \, P + P \, Q^2)^2 - 2 \, \hbar^2 \, Q^2. \quad (14)
\]

3. Discussion of Asymptotic Equality

In order to compare the statistical weight \( w_j \) introduced in (2) with the energy shell volume \( \mathrm{d}m(H) \), we have to translate the quantum mechanical eigen value problem into phase space language. The energy eigen value problem reads

\[
H \, E_j = H_j \, E_j, \quad (15)
\]

where \( H_j \) are the (real) eigen values of the (self-adjoint) Hamilton operator \( H \), and \( E_j \) the projection operators on the corresponding eigen spaces:

\[
E_j^* = E_j, \quad E_j^* = E_j, \quad \text{tr}(E_j) = w_j; \quad (16)
\]

by definition, \( w_j \) are the dimensions of the eigen spaces, hence equal to the traces of the projection operators. Equation (15) can be decomposed into its self-adjoint and anti-self-adjoint part:

\[
[H, E_j] = 0, \quad H \, E_j + E_j \, H = 2 \, H_j \, E_j. \quad (17)
\]

Under the Weyl–Wigner correspondence described in section 2, Eqs. (17) map into the following phase space equations [compare (10), (11)]

\[
[H, E_j] = 0, \quad (18)
\]

and

\[
\cos\left(\frac{1}{2} \, \hbar \, \Gamma\right) \, H(p, q) \, E_j(p, q) = H_j \, E_j(p, q). \quad (19)
\]

Here \( H(p, q) \) is (as always) the Hamilton function, and \( E_j(p, q) \) is the phase space image of the projection operator \( E_j \). If in Eq. (18) the Moyal bracket is replaced by the Poisson bracket, this equation says that \( E_j \) must be a constant of the motion. In the simplest case of the one-dim harmonic oscillator,
this replacement is admissible, and $E_j$ must be a function of $H$. In this case, Eq. (19) becomes an ordinary differential equation for $E_j(H)$.

Let us assume that the operator $H$ has a pure point spectrum. In this case, the spectral theorem says that $\sum_{j=\infty}^{\infty} E_j$ converges weakly (in the trace norm) towards the unit operator for $J \rightarrow \infty$, which implies that $\sum_{j=\infty}^{\infty} E_j$ converges weakly (in the $L^2$ norm) towards the unit function on phase space. We have [compare (8)]

$$ w_j = \{ \langle E_j, 1 \rangle = \langle E_j, 1 \rangle \}, \text{ and}$$

$$ \langle E_j, E_i \rangle = \langle E_j, E_i \rangle, \quad (20) $$

which shows that $E_j$ is a square integrable function for $w_j < \infty$ whose integral is equal to $w_j$.

On the other hand, the measure $dm = dH \, m'(H)$ of the energy shell of thickness $dH$ can be written as

$$ dm = \langle \chi_{\Delta H}, 1 \rangle \quad (21)$$

where $\chi_{\Delta H}(p, q)$ is the characteristic function of the shell. Consequently

$$ d_jm - w_j = \langle \chi_j - E_j, 1 \rangle, \quad (22) $$

so that asymptotic equality of the statistical weight $w_j$ and the shell volume $d_jm$ is guaranteed if there exists a shell decomposition of phase space such that $E_j$ converges (in the $L^1$ norm) towards the characteristic function $\chi_j$ of the $j$-th shell (around $H_j$) for $H_j \rightarrow \infty$.

Such a convergence actually occurs for physically reasonable Hamilton functions, for the following reason: We have already seen that $\{E_j\}$ is a decomposition of the unit function; that is $\sum_j E_j = 1$.

If moreover the function $E_j$ is essentially supported by a shell of decreasing thickness around $H(p, q) = H_j$ (i.e. if it is negligibly small outside of such a shell), this decomposition is essentially a shell decomposition with centers $H_j$.

The support property just mentioned is a consequence of Eq. (19). In order to get an intuitive understanding, let us consider the harmonic oscillator for which $\Gamma^2 H \Delta = 0$ so that (19) simplifies to

$$ \frac{1}{2} \hbar^2 H^{-1} \Gamma^2 H E_j = (1 - H_j/H) E_j. \quad (23) $$

$H$ is by assumption a quadratic form, which implies that the left hand side is (in suitable coordinates) equal to the Laplacian applied to $E_j$. The parenthesis on the right hand side assumes large negative values for $H < H_j$, and tends towards 1 for $H > H_j$. As a consequence, $E_j$ oscillates strongly for $H < H_j$, becomes large for $H \approx H_j$, and decreases exponentially for $H > H_j$. In the limit $H_j \rightarrow \infty$, the essential support of $E_j$ shrinks to the sphere $H = H_j$.

We do not attempt to derive quantitative statements about the eigenvalue distribution $w_j$; for Hamilton operators of the general form

$$ H = \hat{P} F(\hat{Q}) \hat{P} + G(\hat{Q}) $$

this has been done in $^2$. Instead we want to show that without any proviso there are obvious counter examples to an asymptotic convergence. For instance it can happen that the Hamilton operator $H$ has a continuous (part of the) spectrum so that the place of the spectral operators $E_j$ is taken by projection operators on infinite dimensional subspaces. The corresponding statistical heights $w_j$ are infinite. One might conjecture that in this case the volume of the energy shell was likewise infinite, in agreement with the fact that continuous spectra occur in dissociated (ionized) systems whose orbits (in phase space) extend to infinity. However there are Hamilton functions with bounded energy surfaces whose associated operators have a continuous spectrum! Similarly one might conjecture that a positive function gave rise to an operator bounded from below, or that a positive operator gave rise to a function bounded from below. The last conjecture is refuted by the hydrogen atom, and the former one will be likewise disproven.

To this end consider the function

$$ H = (p \, q)^2 + \alpha (p^2 + q^2), \quad \alpha \geq 0. \quad (24) $$

$H$ is evidently positive, and the surfaces $H = \text{const}$ are bounded for every $\alpha > 0$. I am going to prove that its corresponding operator $H$ is unbounded to either side for sufficiently small $\alpha$, and has a continuous spectrum for $\alpha < 7 \hbar^2/4$. First of all, from Eq. (14) we have

$$ H : = \Phi H = (Q^2 + \alpha) P^2 - 4 i \hbar \, Q \cdot P + (\alpha - 3 \hbar^2) Q^2 $$

$$ = \frac{1}{2} (Q^2 \cdot P + P \cdot Q)^2 + \alpha (P^2 - (2 \hbar^2 - 2 \hbar^2) Q^2. \quad (25) $$

The second line shows that $H$ is the difference of two positive unbounded selfadjoint operators, which hints at non-boundedness. Let us evaluate the diagonal matrix element $\langle f, H \rangle$ in the position representation where $f = f(q)$, $Q$ acts as multiplication by $q$.

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and $P$ as differentiation $-i\hbar \partial_j$, ($\partial_j := \partial_j$); and let us content ourselves with the special case $\alpha = 0$

$$\hbar^{-2}(f, H f) = \frac{1}{2} (q^2 \partial_j + \partial_j q^2) f \|f\|^2 - 2 \|q f\|^2$$

$$= \int dq q^2 \left( q f' + f'^2 - 2 |f|^2 \right). \tag{26}$$

It is not difficult to see that there are functions $f(q)$ of norm 1 for which the integral in (26) assumes arbitrarily large positive or negative values: In the first case one has to choose $|f'|$ large for large $|q|$ whereas in the second case one may choose $f$ like a table mountain profile.

In order to see that $H$ in (25) has continuous spectrum for $\alpha < \frac{7}{4} \hbar^2$, we consider its eigenvalue equation in the position representation

$$0 = (H - H_j) f_j$$

which becomes asymptotically for large $|q|$:

$$0 \cong \left( \partial^2 + 4 q^{-2} \Omega + 3 q^{-2} (1 - \alpha/3 \hbar^2) \right) f_j \tag{27}$$

Here we have written $f$ for $f_j$ because the eigenvalue $H_j$ does not enter into the asymptotic form. Eq. (28) can be solved in closed form by means of a power ansatz:

$$f \cong c q^\gamma \text{ with } \gamma = -\frac{3}{2} \pm \sqrt{\frac{\alpha}{\hbar^2} - \frac{3}{4}}. \tag{29}$$

This is the asymptotic form of all eigen functions for $|q| \to \infty$, where the two signs of the square root correspond to the two linearly independent solutions. If both of them are square integrable, so are all solutions of Eq. (27), and the spectrum contains all the reals. This happens for $\gamma < -1/2$, or $\alpha/\hbar^2 < 7/4$ as has been claimed above.

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**Erweiterte Gravitationstheorie, Machisches Prinzip und rotierende Massen**

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*Herrn Professor Dr. PASCUAL JORDAN zum 65. Geburtstag gewidmet*

The rotation of the local inertial frames induced by a rotating shell of mass is calculated in the framework of P. Jordan's "extended theory of gravitation". As a special case, the corresponding results are obtained for Brans and Dicke's "scalar-tensor" theory. In the weak field approximation the result is the same as the Lense-Thirring effect of General Relativity, except for a factor depending on the coupling constant of the scalar field. The strong field limit in which the inertial frame is completely dragged along by the rotating shell is investigated. In particular it is shown that this limit of perfect dragging occurs in certain cosmological models whenever the density of the rest of the matter in the universe tends to zero. This result is interpreted as a manifestation of Mach's principle in the extended theory of gravitation.

Das wachsende Interesse an der Jordanschen erweiterten Gravitationstheorie beruht einerseits auf einer Anzahl experimenteller Tatsachen, die auf eine zeitlich und räumlich veränderliche Gravitationskonstante hinweisen und andererseits auf den anschließenden mathematischen mathematischen Matheseiner

