Nonlinear Incoherent Light Scattering

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Classical equations are used to calculate nonlinear incoherent light scattered from two coherent beams of electromagnetic radiation of different frequencies propagating in a homogeneous plasma without magnetic field. With a suitable choice of the difference frequency, enhanced light scattering occurs near the plasma frequency. In this case the frequency spectrum is mainly given by the thermal ion density fluctuations.

Nonlinear processes, occurring when one or more electromagnetic waves propagate in a collisionless plasma, have been examined both theoretically and experimentally. Theoretical work is of two kinds. In the first type, the plasma is treated using a collective model, be it the Vlasov equation or macroscopic equations. This produces nonlinear waves with frequencies, wave numbers and directions uniquely determined by the incoming (primary) waves because of energy and momentum conservation. (Though, as a result of refraction, a finite plasma volume makes a finite width of the angle of propagation.) In the second type the discrete plasma structure is considered explicitly, which leads to new effects. Each particle separately can exchange energy and momentum with the waves. Since, moreover, all particles interact via Coulomb forces, this presents an intricate problem which to the author's knowledge has been treated hitherto uniquely determined by the incoming (primary) waves. In this case the frequency spectrum is mainly given by the thermal ion density fluctuations.

The nonlinear interaction of electromagnetic waves will be specified here in the following way: Two electromagnetic waves having the frequencies ω₀, σ = 1, 2 the directions \( \mathbf{n} \), and arbitrary polarisation propagate in a homogeneous isotropic plasma: What are the nonlinear (quadratic) effects that can be seen outside the plasma?

1. Basic Equations

The work contains a purely classical description of nonlinear wave phenomena in which the particle aspects of the plasma are included, and the difficulties involved in quantum field theoretical representations are avoided.

In section 1 the problem is specified and the equations for handling it are given. Sections 2 and 3 give a brief summary of results relating to thermal plasmas and linear wave effects. In section 4 nonlinear effects are derived, which are discussed in sections 5 and 6.

### 1. Basic Equations

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Let the primary waves be
\[ E_p(r, t) = \sum_{\alpha} E_{\alpha} \exp\{-i(\mathbf{k}_\alpha \cdot \mathbf{r} - \omega_{\alpha} t)\}; \quad \mathbf{k}_\alpha \cdot E_{\alpha} = 0; \]
\[ B_p(r, t) = \sum_{\alpha} B_{\alpha} \exp\{-i(\mathbf{k}_\alpha \cdot \mathbf{r} - \omega_{\alpha} t)\}; \quad B_{\alpha} = \frac{c}{c} [\mathbf{k}_\alpha \times E_{\alpha}]; \]
\[ \sigma = \pm 1, \pm 2; \quad E_{-\sigma} = E_{\sigma}^*; \quad k_{-\sigma} = -k_{\sigma}; \quad \omega_{-\sigma} = -\omega_{\sigma}; \quad n_{-\sigma} = n_{\sigma}; \quad \mathbf{k}_\sigma = n_{\sigma} \mathbf{k}_0. \]
The plasma consists of \( N_e \) electrons and \( N_i \) ions with charges \( q_e, q_i \) and masses \( m_e, m_i \) in the volume \( V \) and is described by the functions \( F_{\alpha}(r, v, t) \), \( \alpha = e, i \):
\[ F_{\alpha}(r, v, t) = \sum_{i=1}^{N_{\alpha}} \delta(r - \mathbf{r}_i) \delta(v - \mathbf{v}_i) \]
where \( \mathbf{r}_i, \mathbf{v}_i \) are the position and velocity of the \( i \)-th particle. For \( F_{\alpha} \) we have
\[ \frac{\partial F_{\alpha}}{\partial t} + \mathbf{v} \cdot \nabla F_{\alpha} + \frac{q_{\alpha}}{m_{\alpha}} \left( \mathbf{E} + \frac{1}{c} [\mathbf{v} \times \mathbf{B}] \right) \cdot \nabla_v F_{\alpha} = 0 \]
where \( \mathbf{E}(r), \mathbf{B}(r) \) is the effective self-consistent field at the point \( r \). \( F_{\alpha}(r, v, t) \) depends on the values of the particle coordinates at some initial time, say \( t = 0 \). It is useful to consider quantities averaged over the initial values of
\[ \mathbf{r}_i(t = 0) = \mathbf{R}_i; \quad \mathbf{v}_i(t = 0) = \mathbf{v}_i. \]
With such an averaging procedure we put
\[ \langle F_{\alpha}(r, v, t) \rangle = f_{\alpha}(r, v, t), \quad \langle \mathbf{E} \rangle = \mathbf{E}, \quad \langle \mathbf{B} \rangle = \mathbf{B}. \]
The particle structure of the plasma is then contained in the fluctuations
\[ \delta f_{\alpha}(r, v, t) = F_{\alpha} - \langle F_{\alpha} \rangle, \quad \delta \mathbf{E} = \mathbf{E} - \langle \mathbf{E} \rangle, \quad \delta \mathbf{B} = \mathbf{B} - \langle \mathbf{B} \rangle. \]
Averaging (3) there results
\[ \frac{\partial f_{\alpha}}{\partial t} + \mathbf{v} \cdot \nabla f_{\alpha} + \frac{q_{\alpha}}{m_{\alpha}} \mathbf{A} \cdot \nabla_v f_{\alpha} = - \frac{q_{\alpha}}{m_{\alpha}} \left( \delta \mathbf{A} \cdot \nabla_v f_{\alpha} \right), \]
\[ \frac{\partial \delta f_{\alpha}}{\partial t} + \mathbf{v} \cdot \nabla \delta f_{\alpha} + \frac{q_{\alpha}}{m_{\alpha}} \mathbf{A} \cdot \nabla_v \delta f_{\alpha} + \frac{q_{\alpha}}{m_{\alpha}} \delta \mathbf{A} \cdot \nabla_v f_{\alpha} = - \frac{q_{\alpha}}{m_{\alpha}} \left( \delta \mathbf{A} \cdot \nabla_v f_{\alpha} - \langle \delta \mathbf{A} \cdot \nabla_v \delta f_{\alpha} \rangle \right) \]
with
\[ \mathbf{A} = \mathbf{E} + \frac{1}{c} [\mathbf{v} \times \mathbf{B}], \quad \delta \mathbf{A} = \delta \mathbf{E} + \frac{1}{c} [\mathbf{v} \times \delta \mathbf{B}]. \]
(7) is the Vlasov equation with a collisional term. (8) is an equation for the fluctuations. For further simplification the following assumptions are made.

a) Correlations involving more than two particles are neglected, which can be done when there are many particles in a Debye sphere. The right hand side of (8) is thus neglected.

b) Coulomb collisions are assumed to be unimportant, and so the right hand side of (7) is set equal to zero.

c) For the plasma without electromagnetic waves we have
\[ f_{\alpha}^{(0)}(v) = \frac{N_{\alpha}}{V} \left( \frac{1}{\pi^2 v_{\alpha}^2} \right)^{\frac{3}{2}} \exp\{- (v/v_{\alpha})^2\} = n_{\alpha} f_{\alpha 0}(v), \quad v_{\alpha}^2 = \frac{2 T_{\alpha}}{m_{\alpha}}. \]
d) The quantities of interest \( f_{\alpha}, \delta f_{\alpha}, \mathbf{E}, \delta \mathbf{E} \) etc. may be expanded in powers of the amplitude of the primary waves.
\[ f_{\alpha}(r, v, t) = f_{\alpha}^{(0)} + f_{\alpha}^{(1)} + f_{\alpha}^{(2)} + \ldots \]
\[ 10 \text{ Yu. L. Klimontovitch and V. P. Silin, Soviet Phys.-JETP 15, 199 [1962].} \]
\[ 11 \text{ J. M. Dawson and T. Nakayama, Phys. Fluids 9, 252 [1966].} \]
\[ 12 \text{ A. Salat, IPP 6/49 [1966].} \]
Nonlinear effects can thus be found in the second order. It is useful to go over to complex Fourier representation in the following way:

\[
\begin{align*}
    &f_a(r, v, t) = \left( \frac{2\pi}{\omega} \right)^4 \int \int d\omega d^3k \exp\left\{ -i(\mathbf{k} \cdot \mathbf{r} - \omega t) \right\} f_a(\omega, \mathbf{k}, \mathbf{v}), \\
    &f_a(\omega, \mathbf{k}, \mathbf{v}) = \int dt \int d^3r \exp\left\{ i(\mathbf{k} \cdot \mathbf{r} - \omega t) \right\} f_a(\mathbf{r}, \mathbf{v}, t).
\end{align*}
\]

(12)

The asterisk indicates that the path of integration in the complex \( \omega \)-plane goes below all singularities of the integrand. With a) and b) there results

\[
\begin{align*}
    &i(\omega - \mathbf{k} \cdot \mathbf{v}) \, \delta f_a(\omega, \mathbf{k}, \mathbf{v}) - f_a(t = 0, \mathbf{k}, \mathbf{v}) + \frac{q_a}{m_a} \mathbf{A} \cdot \nabla \delta f_a = 0, \\
    &i(\omega - \mathbf{k} \cdot \mathbf{v}) \, f_a(\omega, \mathbf{k}, \mathbf{v}) - \delta f_a(t = 0, \mathbf{k}, \mathbf{v}) + \frac{q_a}{m_a} \mathbf{A} \cdot \nabla \delta f_a + \frac{q_a}{m_a} \mathbf{A} \cdot \nabla \delta f_a = 0
\end{align*}
\]

(13)

(14)

where \( \mathbf{A} \circ \mathbf{B} \) stands for the convolution integral \( (2\pi)^{-4} \int d\omega' \int d^3k' A(\omega', k') \cdot B(\omega - \omega', \mathbf{k} - \mathbf{k}') \). In Fourier representation Maxwell’s equations can be put in the following form:

\[
\begin{align*}
    &\mathbf{E}(\omega, \mathbf{k}) = \frac{E_0(\omega, \mathbf{k})}{k^2 \epsilon^2 - \omega^2} + \frac{4\pi i}{k^2} \left\{ \frac{\mathbf{q}(\omega, \mathbf{k})}{k^2 \epsilon^2 - \omega^2} \left[ \mathbf{k} \times \left[ \mathbf{k} \times \mathbf{E}(t = 0, \mathbf{k}) \right] \right] \right\}, \\
    &\mathbf{E}(\omega, \mathbf{k}) = \frac{-i}{k^2} \left\{ \omega \left[ \mathbf{k} \times \left[ \mathbf{k} \times \mathbf{E}(t = 0, \mathbf{k}) \right] \right] + k^2 \epsilon \left[ \mathbf{k} \times \mathbf{B}(t = 0, \mathbf{k}) \right] \right\}, \quad k = |\mathbf{k}|
\end{align*}
\]

(15)

(16)

and analogously for the fluctuations.

By specifying initial conditions the system (13) – (17) is fully determined.

2. Thermal Plasma

This well-known regime is presented briefly here because the following sections depend closely on the way solutions are found in this section.

After separating into longitudinal and transverse parts the Eqs. (13) – (15) read:

\[
\begin{align*}
    &\mathbf{E}^{(0)}(\omega, \mathbf{k}) = 0, \quad i(\omega - \mathbf{k} \cdot \mathbf{v}) \, \delta f_a^{(0)}(\omega, \mathbf{k}, \mathbf{v}) + \frac{q_a}{m_a} \mathbf{A} \cdot \nabla \delta f_a^{(0)}(t = 0, \mathbf{k}, \mathbf{v}) = 0, \\
    &\mathbf{k} \cdot \mathbf{\delta E}^{(0)}(\omega, \mathbf{k}) = 4\pi q^{(0)}(\omega, \mathbf{k}), \quad \left[ \mathbf{k} \times \mathbf{\delta E}^{(0)}(\omega, \mathbf{k}) \right] = \frac{\mathbf{k} \times \left[ \mathbf{k} \times \mathbf{E}(t = 0, \mathbf{k}) \right]}{k^2 \epsilon^2 - \omega^2} - \frac{4\pi i \omega}{k^2 \epsilon^2 - \omega^2} \left[ \mathbf{k} \times \mathbf{j}^{(0)}(\omega, \mathbf{k}) \right].
\end{align*}
\]

(18)

(19)

(20)

Taking moments from (19), (20) there results

\[
\begin{align*}
    &\mathbf{k} \cdot \mathbf{\delta E}^{(0)}(\omega, \mathbf{k}) = 4\pi \sum q_a \int \frac{d^3v}{\omega - \mathbf{k} \cdot \mathbf{v}} \left\{ \delta f_a^{(0)}(t = 0, \mathbf{k}, \mathbf{v}) - \frac{q_a}{m_a} \mathbf{A} \cdot \nabla \delta f_a^{(0)}(t = 0, \mathbf{k}, \mathbf{v}) \right\}, \\
    &\mathbf{E}(\omega, \mathbf{k}) = \sum \mathbf{k} \cdot \mathbf{\delta E}^{(0)} = \sum \frac{4\pi}{\epsilon(\omega, \mathbf{k})} q_a \int \frac{d^3v}{\omega - \mathbf{k} \cdot \mathbf{v}} \delta f_a^{(0)}(t = 0, \mathbf{k}, \mathbf{v})
\end{align*}
\]

(21)

(22)

and from this

\[
\begin{align*}
    &\mathbf{E}(\omega, \mathbf{k}) = \frac{4\pi \sum q_a}{\epsilon(\omega, \mathbf{k})} \int \frac{d^3v}{\omega - \mathbf{k} \cdot \mathbf{v}} \delta f_a^{(0)}(t = 0, \mathbf{k}, \mathbf{v}), \\
    &\mathbf{E}(\omega, \mathbf{k}) = -\frac{4\pi q_a^2}{m_a} \mathbf{A}\cdot \mathbf{k} \cdot \mathbf{E}^{(0)}(\omega, \mathbf{k})
\end{align*}
\]

(23)

For the approximations to the longitudinal dielectric constant \( \epsilon(\omega, \mathbf{k}) \) see Appendix A.

Analogously, from taking the velocity moment it follows that

\[
\begin{align*}
    &\left[ \mathbf{k} \times \mathbf{\delta E}^{(0)}(\omega, \mathbf{k}) \right] = \frac{\left[ \mathbf{k} \times \mathbf{\delta E}^{(0)} \right]}{2(\omega, \mathbf{k})} + \sum q_a \left[ \mathbf{k} \times \mathbf{\delta E}^{(0)}(\omega, \mathbf{k}) \right] \\
    &\left[ \mathbf{k} \times \mathbf{\delta E}^{(0)} \right] = \frac{\left[ \mathbf{k} \times \mathbf{\delta E}^{(0)} \right]}{2(\omega, \mathbf{k})} + \sum q_a \int \frac{d^3v \, \mathbf{A} \cdot \mathbf{k} \times \mathbf{E}^{(0)}(t = 0, \mathbf{k}, \mathbf{v})}{\omega - \mathbf{k} \cdot \mathbf{v}} \delta f_a^{(0)}(t = 0, \mathbf{k}, \mathbf{v})
\end{align*}
\]

(24)
with
\[
\alpha(\omega, k) = k^2 c^2 - \omega^2 + \sum_{\alpha} \omega_{\alpha}^2 \int_{-\infty}^{+\infty} \frac{dv}{\omega - k \cdot v} f_0(v).
\]

Again, taking moments of (19), the following density and velocity fluctuations occur:
\[
\delta N_2^{(0)}(\omega, k) = \frac{1}{4\pi i q_a} \sum_{\beta} \delta E_{\beta}^{(0)}(\omega, k) a_{\alpha\beta}(\omega, k),
\]
\[
k \cdot \delta V_2^{(0)}(\omega, k) = \frac{\omega}{4\pi i q_a} \sum_{\beta} [k \cdot \delta E_{\beta}^{(0)}] a_{\alpha\beta}(\omega, k) + i \int d^3v \delta f_2^{(0)}(t = 0, k, v),
\]
\[
[k \times \delta V_2^{(0)}(\omega, k)] = -\frac{1}{4\pi i \omega q_a} \sum_{\beta} [k \times \delta E_{\beta}^{(0)}] b_{\alpha\beta}(\omega k) - \frac{i q_a}{m_a} \frac{k \times \delta E_2^{(0)}}{\omega - k \cdot v} \int \frac{dv}{\omega - k \cdot v}
\]
with
\[
a_{\alpha\beta}(\omega, k) = e^{(\omega, k)} \delta_{\alpha\beta} + \omega_a^2 \int \frac{dv f_0(v)}{(\omega - k \cdot v)^2}, \quad b_{\alpha\beta}(\omega, k) = \alpha(\omega, k) \delta_{\alpha\beta} - \omega_a^2 \int \frac{dv f_0(v)}{\omega - k \cdot v}.
\]

The well-known formula for thermal density fluctuations follows from (26), (22) by averaging over the initial values (4) of \( \mathbf{r}_i, \dot{\mathbf{r}}_i \). Using
\[
\lim_{\gamma \to 0} \gamma (|\delta N_2^{(0)}(\omega, k)|^2) = \frac{1}{V} \lim_{\gamma \to 0} \gamma \left( \sum_{\alpha} \frac{N_2^{(0)}}{\omega - k \cdot v} f_0(\omega) f_0(\omega) \right) = \frac{1}{V} \lim_{\gamma \to 0} \gamma \left( \sum_{\alpha} f_0(\omega) f_0(\omega) \right)
\]
In the following, the transverse part of thermal fluctuations is neglected.

### 3. Linear Processes

From the linearized Vlasov Eq. (13) the frequencies \( \omega_\alpha \) and wave numbers \( k_\alpha \) of waves (1) are coupled by the dispersion relation
\[
\alpha(\omega_\alpha, k_\alpha) = k_\alpha^2 c^2 - \omega_\alpha^2 + \sum_{\alpha} \omega_{\alpha}^2 = 0.
\]

The plasma together with the waves oscillates according to
\[
E_{(1)}^{(1)}(\omega, k) = \sum_{\alpha} E_{\alpha} \frac{-i(2\pi)^3 \delta(k - k_{\alpha})}{\omega - \omega_{\alpha}},
\]
\[
f_{(1)}^{(1)}(\omega, k, v) = \sum_{\alpha} f_{\alpha} \frac{-i(2\pi)^3 \delta(k - k_{\alpha})}{\omega - \omega_{\alpha}}, \quad f_{\alpha} = \frac{i q_a}{m_a \omega} E_{\alpha} \cdot \nabla v f_2^{(0)}(v),
\]
\[
N_{(1)}^{(1)}(\omega, k) = \int d^3v f_{(1)}^{(1)}(\omega, k, v) \equiv 0
\]
results which follow trivially from (13), (15), (16) and the assumption \(|\omega/k \cdot v| \gg 1\), which for \( \omega = \omega_\alpha, k = k_\alpha \) and (32) is automatically fulfilled.

Besides these collective effects, the primary waves produce fluctuations in the plasma.

The linear approximation to Eq. (14) reads
\[
i(\omega - k \cdot v) \delta f_2^{(1)}(\omega, k, v) + \frac{q_a}{m_a} \delta E_{(1)}^{(1)} \cdot \nabla v f_2^{(0)} = \delta f_2^{(1)}(t = 0) - \frac{q_a}{m_a} \mathbf{A}_{(1)} \cdot \nabla v f_2^{(0)} - \frac{q_a}{m_a} \delta E_{(0)} \cdot \nabla v f_2^{(1)}.
\]

In comparison to the thermal case, only the inhomogeneous right hand side has changed. Since also Maxwell's equations are the same in any order, the solution to Eq. (34) can immediately be written down by comparing with the last section. Choosing \( \delta f_2^{(1)}(t = 0) = \delta E_{(1)}^{(1)}(t = 0) = 0 \), omitting the B-field of the waves and unfolding the convolution integral, there results with the help of (22) and (24)
\[
k \cdot \delta E_{(1)}^{(1)}(\omega, k) = \int d^3v \frac{m_a}{k \cdot \mathbf{E}_{(0)}} \frac{q_a}{m_a} \delta E_{(0)}(\omega - \omega_\alpha, k - k_{\alpha}, v) + f_2 \delta E_{(0)}(\omega - \omega_\alpha, k - k_{\alpha}),
\]

(35)
\[
\begin{align*}
[k \times \delta E^{(1)}(\omega, k)] &= - \frac{4\pi}{\varepsilon(\omega, k)} \sum q_a^2 \int \frac{dv \omega [k \times v]}{(\omega - k)^2} \left( E_\omega \delta f'_0(\omega - \omega_a, k - k_a) \right. \\
&\quad + f_{aa} \delta E^{(0)}(\omega - \omega_a, k - k_a) \bigg) - \frac{4\pi}{\varepsilon(\omega, k)} \sum q_a^2 \int \frac{dv \omega}{\omega - k} \\
&\quad \left. \cdot \left[ k \times \left( E_\omega \delta f'_0(\omega - \omega_a, k - k_a) + f_{aa} \delta E^{(0)}(\omega - \omega_a, k - k_a) \right) \right] \right].
\end{align*}
\]

(35), (36) considerably simplify under the restriction \(|\omega/|k \cdot v| \gg 1\). Then the contribution of the pole \(k \cdot v = 0\) is exponentially small, and from the expansion of the denominator there results
\[
\begin{align*}
[k \times \delta E^{(1)}(\omega, k)] &= \sum q_a^2 k \cdot \delta E^{(1)}_\omega = \frac{4\pi}{\varepsilon(\omega, k)} \sum q_a^2 \left\{ k \cdot E_\omega \delta f'_0(\omega - \omega_a, k - k_a) \right. \\
&\quad + f_{aa} \delta E^{(0)}(\omega - \omega_a, k - k_a) \bigg) + \frac{\varepsilon}{\varepsilon(\omega, k)} k \cdot E_\omega \delta f'_0(\omega - \omega_a, k - k_a) + f_{aa} \delta E^{(0)}(\omega - \omega_a, k - k_a) \right].
\end{align*}
\]

(38) is the well-known formula for the scattering of electromagnetic waves on density fluctuations in a plasma, written in \(k\)-space. The second term describes scattering of thermal fields on induced plasma polarization. Since the poles of \(\delta N^{(0)}(\omega - \omega_a, k - k_a)\) are close to \(\omega = \omega_a\), this term is smaller than the first by a factor of \(\omega_a/\omega_0\) and is generally omitted for \(\omega_0^2 \gg \omega_a^2\).

The connection between induced density etc. fluctuations and fluctuations of the electrical field because of the analogous form of the equations is the same as in the thermal case:
\[
\delta N^{(1)}_{\omega a}(\omega, k) = \frac{1}{4\pi i} q_a \sum q_a^2 k \cdot \delta E^{(1)}_{\omega a}(\omega, k) a_{\omega a}(\omega, k) \quad (40)
\]

etc. as in (27), (28).

4. Nonlinear Processes

Nonlinear collective processes, e.g. periodical density changes induced by two electro-magnetic waves were discussed in 5, 6, 8. There was proposed to detect these density variations by aiming a third beam at them and looking for the light which is "scattered" into very special angles only. It is therefore sufficient here to summarize very briefly the collective processes. Eq. (13) reads for \(f^{(2)}_x(t = 0) = 0\)
\[
i(\omega - k \cdot v) f^{(2)}_x(\omega, k, v) + \frac{q_a}{m_a^*} E^{(2)} \cdot \nabla v f^{(0)}_x = - \frac{q_a}{m_a^*} A^{(1)} \circ \nabla v f^{(1)}_x.
\]

Again by analogy with section 2, the solution is
\[
k \cdot E^{(2)}(\omega, k) = - \frac{4\pi}{\varepsilon(\omega, k)} \sum q_a^2 \int \frac{dv \omega}{\omega - k} \left( E^{(1)} + \frac{1}{\varepsilon} [v \times B^{(1)}] \right) \circ \nabla v f^{(1)}_x.
\]

For \(|\omega/|k \cdot v| \gg 1\) i.e. \(|\omega_a/|k \cdot v| \gg 1\) (43) the expansion gives
\[
E^{(2)}(\omega, k) = \frac{\sum q_a^2 E_{x, a}}{\omega - \omega_a^2} \delta (k - k_a), \quad E_{x, a} = - \frac{i q_a}{m_a} \frac{\omega_a^2}{2 \omega_a \omega_x} \frac{1}{\omega (\omega_a, k_a) k_a E_{x} \cdot E_{t}} \quad (44)
\]

with \(\omega_a = \omega_a + \omega_t\), \(k_a = k_a + k_t\), \(k_a = |k_a|\) (45).

The transverse part of \(E^{(2)}\) vanishes identically. The amplitude of \(E^{(2)}\) may become large, when the difference frequency of \(\omega_a\) is close to the plasma frequency \(\omega_p = (\sum \omega_a)^{1/2}\).

\[12\] E. F. SALPETER, Phys. Rev. 120, 1528 [1960].
For the nonlinear fluctuations (14) gives
\[ i(\omega - k \cdot v) \delta f^{(2)}_\omega (\omega, k, v) + \frac{q_n}{m_a} \delta E^{(2)}(\omega) \cdot \nabla f^{(1)}_\omega \]
\[ = \delta f^{(2)}_\omega (t = 0, k, v) - \frac{q_n}{m_a} \left\{ (A^{(1)} \cdot \nabla f^{(1)}_\omega + \delta A^{(1)} \cdot \nabla f^{(1)}_\omega + \nabla E^{(1)} \cdot \nabla f^{(1)}_\omega + \delta E^{(1)} \cdot \nabla f^{(1)}_\omega) \right\}. \] (46)

For \( \delta f^{(2)}_\omega (t = 0) = \delta E^{(2)}(t = 0) = 0 \) and with Maxwell’s equations the transverse field follows from analogy with the thermal regime (24):
\[ [k \times \delta E^{(2)}(\omega, k)] = - \frac{4 \pi}{\omega_{\infty}(k, k)} \sum \frac{q_n}{m_a} \int \frac{d^2 \omega [k \cdot v]}{\omega - k \cdot v} \cdot \left\{ (A^{(1)} \cdot \nabla f^{(1)}_\omega + \delta A^{(1)} \cdot \nabla f^{(1)}_\omega + \nabla E^{(1)} \cdot \nabla f^{(1)}_\omega + \delta E^{(1)} \cdot \nabla f^{(1)}_\omega) \right\}. \] (47)

As can be seen by the pole \( \alpha(\omega, k) = 0 \) (47) contains fluctuating electric (electromagnetic) fields that propagate as waves in the plasma, and that are radiated away when the plasma volume is finite. This nonlinearly scattered light comes from four different processes: a) Primary waves \( A^{(1)} \) are scattered by induced fluctuations \( \delta f^{(1)}_\omega \). b) Induced fluctuations of the electric field \( \delta A^{(1)} \) are scattered by the plasma polarisation \( f^{(1)}_\omega \). c) Nonlinear fields \( E^{(1)} \) are scattered by thermal fluctuations \( \delta \omega^{(0)} \). d) Thermal fields are scattered by the polarization \( f^{(1)}_\omega \). The scattered light, (47), due to the stochastic fluctuation, has a finite time and length of coherence. It may be called partially coherent or incoherent.

Since \( \alpha(\omega, k) = 0 \) for real \( \omega \) only permits \( |\omega/k| \geq c \), the denominator of (47) can be expanded to give
\[ [k \times \delta E^{(2)}(\omega, k)] = - \frac{4 \pi}{\omega_{\infty}(k, k)} \sum \frac{q_n}{m_a} \int \frac{d^2 \omega [k \cdot v]}{\omega - k \cdot v} \cdot \left\{ (k \times E_a) \delta N^{(1)}_a (\omega - \omega_a) + \omega [k \cdot E_a] (k \times V_{a, a}) \right\}
+ \frac{1}{\omega} \left\{ [k \times E_a] k \cdot V_{a, a} + \frac{1}{c} [k \times \delta V^{(1)}_\omega (\omega - \omega_a) \times B_a] \right\}
+ \frac{1}{c} \left\{ [k \times [V_{a, a} \times B^{(1)}(\omega - \omega_a)] + [k \times E_a] \delta N^{(0)}_a (\omega - \omega_a) \right\}
+ [k \times \delta E^{(0)}(\omega - \omega_a) N_{a, a} + \frac{1}{c} k \cdot \delta E^{(0)}(\omega - \omega_a) [k \times V_{a, a}] \right\}
+ \frac{1}{\omega} k \times \delta E^{(0)}(\omega - \omega_a) k \cdot V_{a, a} \right\}. \] (48)

The frequency spectrum of (48) is centered around the frequencies \( \omega_s = \omega_a + \omega_\tau \). Using the results of sections 2 and 3 gives two types of terms:
\[ [k \times \delta E^{(2)}(\omega, k)] = [k \times \delta E_1] + [k \times \delta E_1] \] (49)
with
\[ [k \times \delta E_1] = \frac{4 \pi i}{\omega_{\infty}(k, k)} \sum \frac{q_n}{m_a} \int \frac{d^2 \omega [k \times E_a] \delta N^{(1)}_a (\omega - \omega_a) + \frac{1}{\omega} \left\{ [k \times E_a] k \cdot V_{a, a} + \frac{1}{c} [k \times \delta V^{(1)}_\omega (\omega - \omega_a) \times B_a] \right\}
+ \frac{1}{c} \left\{ [k \times [V_{a, a} \times B^{(1)}(\omega - \omega_a)] + [k \times E_a] \delta N^{(0)}_a (\omega - \omega_a) \right\}
+ [k \times \delta E^{(0)}(\omega - \omega_a) N_{a, a} + \frac{1}{c} k \cdot \delta E^{(0)}(\omega - \omega_a) [k \times V_{a, a}] \right\}
+ \frac{1}{\omega} k \times \delta E^{(0)}(\omega - \omega_a) k \cdot V_{a, a} \right\}. \] (50)
\[ [k \times \delta E_1] = \frac{4 \pi i}{\omega_{\infty}(k, k)} \sum \frac{q_n}{m_a} \int \frac{d^2 \omega [k \times E_a] \delta N^{(1)}_a (\omega - \omega_a) + \frac{1}{\omega} \left\{ [k \times E_a] k \cdot V_{a, a} + \frac{1}{c} [k \times \delta V^{(1)}_\omega (\omega - \omega_a) \times B_a] \right\}
+ \frac{1}{c} \left\{ [k \times [V_{a, a} \times B^{(1)}(\omega - \omega_a)] + [k \times E_a] \delta N^{(0)}_a (\omega - \omega_a) \right\}
+ [k \times \delta E^{(0)}(\omega - \omega_a) N_{a, a} + \frac{1}{c} k \cdot \delta E^{(0)}(\omega - \omega_a) [k \times V_{a, a}] \right\}
+ \frac{1}{\omega} k \times \delta E^{(0)}(\omega - \omega_a) k \cdot V_{a, a} \right\}. \] (51)

The origin of \([k \times \delta E_1]\) are processes of types a) and b). \([k \times \delta E_1]\) comes from processes c) and d).
The fluctuations $\delta E^{(1)}$, $\delta B^{(1)}$, $\delta V^{(1)}$ have been replaced by their longitudinal parts. In the contribution of the transverse parts is shown to be negligible, for $\omega_s^2 > \omega_a^2$. Depending on the value of the frequency $\omega_s$, two different approximations to the scattered field are discussed in the next sections.

5. Nonresonant Case

If $\omega_s$ is not near the plasma frequency $\omega_p$ and if, moreover,

$$\omega_s^2 \gg \omega_a^2$$

only parts of $[k \times \delta E_\perp]$ contribute effectively to the scattering:

$$[k \times \delta E^{(2)}(\omega, k)] = \frac{4 \pi i}{2(\omega, k) \sigma \tau} \frac{1}{\omega_\perp} \left[ [k \times E_e] l_e \cdot E_t \frac{\omega_e}{\omega_\perp} + k \cdot E_e [k \times E_t] \right]$$

$$\times \left\{ \delta N_e^0(\omega - \omega_s) + \left( 1 - \frac{2 k_s^2}{\omega_p^2} \right) \frac{1}{\omega_\perp} \left( q_e \delta N_e^0(\omega - \omega_s) + q_i \delta N_i^0(\omega - \omega_s) \right) \right\}.$$ (52)

Eq. (52) describes the scattering of incident waves at induced density and velocity fluctuations. The energy, emitted from the plasma per unit of time into the solid angle $d\Omega$ and the frequency interval $d\omega$,

$$dI(\omega, n) = \frac{e^2}{4 \pi} \frac{c^2}{m_e c^2} \frac{\omega_s}{\omega_a} \sum_{\sigma \tau} \left[ n \times \left\{ n \times \left( E_e \frac{\omega_e}{\omega_\perp} (n \omega_s - n_\perp \omega_a) \cdot E_t \right) 

+ E_e n \cdot E_e + E_r n \cdot E_r + [E_r \times [n \cdot E_e]] \right\} \right]^2.$$ (53)

is thus found on averaging over the initial conditions of the plasma (see Appendix B) to be:

$$\left\{ n \times \left( E_e \frac{\omega_e}{\omega_\perp} (n \omega_s - n_\perp \omega_a) \cdot E_t \right) 

+ E_e n \cdot E_e + E_r n \cdot E_r + [E_r \times [n \cdot E_e]] \right\} \right]^2.$$ (54)

with

$$\tilde{k} = \frac{\vec{k} \cdot \vec{n}}{\omega_e} \cdot \frac{n}{r} \cdot \vec{n} = r/\tau, \quad \tilde{k} c = (\omega^2 - \omega_p^2)^{1/2} \text{sign } \text{Re } \omega.$$ (55)

The spectrum of the scattered light is given by the electron density fluctuations, as in the linear case. The angular distribution differs from a dipole characteristic. The order of magnitude of the intensity is smaller than the linear one by a factor $\eta$:

$$\eta = (e E_0/m_e \omega_e c)^2.$$ (56)

6. Resonant Case

Let the frequencies $\omega_1, \omega_2$ be such that the difference frequency is close to the plasma frequency:

$$|\omega_s| = |\omega_1 - \omega_2| \approx \omega_p$$

The amplitudes of the collective oscillations $E^{(2)}, N_e^{(2)}, V^{(2)}$ will then be high, and so parts of $[k \times \delta E_\perp]$ dominate in the scattering process. The low frequency wing of the scattered light will not be observable, however, whenever it drops below the plasma frequency. For $m_e \ll m_i$, Eq. (51) becomes

$$[k \times \delta E^{(2)}(\omega, k)] = \frac{4 \pi i}{2(\omega, k) \sigma \tau} \frac{q_e^2}{m_e^2} E_e \cdot E_t \times$$

$$\times \left\{ \delta N_e^0(\omega - \omega_s) + \left( 1 - \frac{2 k_s^2}{\omega_p^2} \right) \frac{1}{q_e} \left( q_e \delta N_e^0(\omega - \omega_s) + q_i \delta N_i^0(\omega - \omega_s) \right) \right\}.$$ (57)

Because of

$$k^2 c^2 \rightarrow k^2 c^2 \ll \omega_p^2 \quad \text{and} \quad k_s^2 c^2 \approx (n_1 \omega_1 - n_2 \omega_2)^2 \approx (\omega_1^2 - \omega_2^2) \omega_p^2 \approx \omega_p^2$$

it follows that

$$\frac{k_s^2}{k^2} \approx 1 + \frac{\omega_1^2 - \omega_2^2}{\omega_p^2} 4 \sin^2 \frac{\omega_1^2}{2} \gg 1.$$ (58)
except for nearly parallel primary beams. \( \alpha_{12} \) is the angle between beam 1 and beam 2. With (60) the electron density fluctuations disappear from (59):

\[
[k \times \delta E^2] = -\frac{4\pi i}{\varepsilon(\omega_s, k_s)} \sum_{\omega_s \omega_s} \frac{q_e^3}{m_e c^2} \left[ \frac{k \times k_s}{2} \right] \delta N^{(0)}(\omega - \omega_s, k - k_s).
\]

The **Pointing vector** gives the energy emitted into the solid angle \( d\Omega \) and the frequency interval \( d\omega \) (see Appendix B)

\[
\langle dI(\omega, \mathbf{n}) \rangle = \left( \frac{\alpha_c^2}{4\pi m_e c^2} \right)^2 \sum_{\omega_s \omega_s} \left| \mathbf{E}_s \cdot \mathbf{E}_r \right|^2 \left( \frac{\varepsilon}{m_e \omega_s c} \right)^2 \left( \frac{\omega_s}{\omega_r} \right)^2 \left( \frac{q_i}{q_0} \right)^2 \sin^2(\mathbf{n}, \mathbf{n}_s) 
\cdot \frac{1}{\varepsilon(\omega_s, k_s)^2} \left( \frac{\omega_s^2 - \omega_p^2}{\omega_p^2} \right)^{\frac{1}{2}} \lim_{\gamma \to 0} \frac{\gamma}{\pi} \langle | \delta N^{(0)}(\omega - \omega_s, \mathbf{k} - \mathbf{k}_s)|^2 \rangle d\omega d\Omega.
\]

The order of magnitude of the scattering cross section is higher than in the nonresonant case by a factor of \((1 - \alpha_c^2/\omega_p^2)^{1/2}\) and “higher” than the linear cross section by \((1 - \alpha_c^2/\omega_p^2)^{1/2} \alpha_c/\omega_c\).

The actual amplitude of the resonance in \( \varepsilon_j \) depends on the **Landau** damping or collisional damping, whichever is the larger. The frequency spectrum of (62) is given by the ion density fluctuations. From (31):

\[
\lim_{\gamma \to 0} \frac{\gamma}{\pi} \langle | \delta N^{(0)}(\Delta \omega, \Delta \mathbf{k})|^2 \rangle = \frac{V}{A_k} \left( \frac{\omega}{2} \int \frac{d\omega_0(v)}{(\Delta \omega - A_k v)^2} \right)^2 
+ n_e \int \frac{d\omega_0(v)}{(\Delta \omega - A_k v)^2} \left( \frac{\omega_0^2}{2} \int \frac{d\omega_0(v)}{(\Delta \omega - A_k v)^2} \right)^2 = S(\Delta \omega, \mathbf{n})
\]

with

\[
\Delta \omega = \omega - \omega_s, \quad \Delta \mathbf{k} = \Delta \mathbf{k}_s = | \mathbf{k} - \mathbf{k}_s |.
\]

It seems appropriate at this stage to discuss briefly the frequency dependence of the ion fluctuations. On the assumption that \( T_e \) and \( T_i \) are not too extremely different the following approximations are found

a) \( \Delta \omega \parallel A_k v_s \ll 1 \), \( \alpha = e, i \)

Using Eq. (A4)

\[
S(\Delta \omega, \mathbf{n}) = \frac{V}{A_k} \left( \frac{1 + \alpha_e^2}{1 + \sum \alpha_e^2} \right)^2 n_i \int_0^{\Delta \omega} \left( \frac{\omega}{A_k} \right) \left( \frac{\omega}{\Delta \omega} \right)^2 n_e \int_0^{\Delta \omega} \left( \frac{\omega}{A_k} \right) \left( \frac{\omega}{\Delta \omega} \right)^2
\]

with

\[
\alpha_e^2 = 1/(\Delta A k^2 \lambda_e^2), \quad \lambda_e^2 = v_e^2/2 \omega_e^2.
\]

Since for small arguments \( f_0(v) \) goes like \( v/2 \), the first term of (65) dominates. Thus for arbitrary \( \alpha_e \) the center of the spectrum has an essentially Gaussian profile corresponding to the ion thermal velocity.

b) \( \Delta \omega \parallel A_k v_s \ll 1 \), \( \Delta \omega \parallel A_k v_i \gg 1 \).

According to Eqs. (A3), (A4)

\[
S(\Delta \omega, \mathbf{n}) = \frac{V}{A_k} \left( \frac{1 + \alpha_e^2}{1 + \sum \alpha_e^2} \right)^2 n_i \int_0^{\Delta \omega} \left( \frac{\omega}{A_k} \right) \left( \frac{\omega}{\Delta \omega} \right)^2 n_e \int_0^{\Delta \omega} \left( \frac{\omega}{A_k} \right) \left( \frac{\omega}{\Delta \omega} \right)^2
\]

with

\[
\mu_a = 1 + \frac{1}{2} (A_k v_s)^2, \quad \Delta A = 2 \sqrt{\frac{\pi}{\alpha}} \sum \omega_s \left( \frac{\omega}{\Delta \omega} \right)^2 \left( \frac{\omega}{A_k v_s} \right)^2 \exp \left( - \frac{\Delta \omega}{A_k v_s} \right)^2.
\]

In this regime there is a more or less pronounced maximum near \( (\Delta \omega)^2 = \omega_e^2 \mu_a(1 + \alpha_e^2)^{-1} \), corresponding to ion oscillations. \( S(\Delta \omega, \mathbf{n}) \) at this frequency is higher than the corresponding electron spectrum by a factor of \((1 + \alpha_e^2)^2/\alpha_e^4 \).

c) \( \Delta \omega \parallel A_k v_s \gg 1 \), \( \alpha = e, i \).

The ion term can be neglected because of its exponential smallness. Thus

\[
S(\Delta \omega, \mathbf{n}) = \frac{V}{A_k} \left( \frac{\omega_i}{\Delta \omega} \right)^2 \left( \frac{\omega_i}{\Delta \omega} \right)^2 n_e \int_0^{\Delta \omega} \left( \frac{\omega}{A_k} \right).
\]

The electron satellite line at the frequencies where the denominator is small is smaller by a factor of \((m_e/m_i)^2 \) than in the electron spectrum and will hardly be visible.
7. Conclusions

In the foregoing sections, a microscopic model has been used to calculate scattered light the amplitude of which depends quadratically on the amplitude of the scattering light. It has been shown that by choosing two light frequencies appropriately resonances give higher intensity for the scattering process. A comparison of the results with those given in $^2$ is suggested. The effects in $^2$ seem to be different, however, since there a second order process is found by combining a linear and a third-order amplitude to give a quadratic effect in the intensity.

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Appendix A

The function

$$Z_a \left( \frac{\omega}{k v_a} \right) = \omega_a^2 \int_{-\infty}^{+\infty} \frac{dv f_\phi(v)}{(\omega - kv)^2} = - \frac{1}{k^2 \lambda_a^2} \left(1 + \frac{\omega}{k v_a} \right) \left( \frac{1}{\pi} \int_{v - \omega/k v_a}^{+\infty} \frac{de^{-v^2}}{v} \right)$$

(A 1)

can be expressed in terms of known functions, see $^{13}$

$$Z_a(x) = - \frac{1}{k^2 \lambda_a^2} \left(1 - i x \sqrt{\pi} e^{-x^2} \left[1 - \text{erf}(ix)\right]\right).$$

(A 2)

For $\lim \text{Im} \omega = 0$ the TAYLOR series or the asymptotic representation gives

for $|\omega|/k v_a \gg 1$: $Z_a \left( \frac{\omega}{k v_a} \right) = - \frac{1}{k^2 \lambda_a^2} \left(1 - i \sqrt{\pi} \omega \frac{\omega}{k v_a} + \ldots\right)$. (A 3)

for $|\omega|/k v_a \ll 1$: $Z_a \left( \frac{\omega}{k v_a} \right) = - \frac{1}{k^2 \lambda_a^2} \left(1 - i \sqrt{\pi} \omega \frac{\omega}{k v_a} + \ldots\right)$. (A 4)

The corresponding formulas for $\varepsilon(\omega, k)$ can be found from

$$\varepsilon(\omega, k) = 1 - \sum Z_a(\omega/k v_a).$$

(A 5)

Appendix B

The integral

$$M_a(\omega, r) = \frac{1}{(2\pi)^3} \int d^3k \exp\left\{-i k \cdot r\right\} \frac{1}{k^2} \delta N_a^{(0)}(\omega - \omega_s, \mathbf{k} - \mathbf{k}_s)$$

(B 1)

is easily calculated with the aid of (26), (2), and (6). For $\omega \neq \omega_s$

$$M_a(\omega, r) = \sum_{i=1}^{N_a} \int_0^\infty dt \exp\left\{-i (\omega - \omega_s) t\right\} \frac{1}{(2\pi)^3} \int \frac{1}{k^2} \delta N_a^{(0)}(\omega - \omega_s, \mathbf{k} - \mathbf{k}_s) \frac{1}{(2\pi)^3} \int \frac{1}{k^2} \exp\left\{-ik \cdot r_i(t)\right\} \exp\left\{-i k \cdot r\right\}$$

(B 2)

$$= \sum_{i=1}^{N_a} \int_0^\infty dt \exp\left\{-i (\omega - \omega_s) t\right\} \exp\left\{-i \mathbf{k}_s \cdot r_i(t)\right\} \frac{1}{(2\pi)^3} \frac{1}{k^2} \delta N_a^{(0)}(\omega - \omega_s, \mathbf{k} - \mathbf{k}_s) \exp\left\{-i \tilde{R}_i^*\right\}$$

(B 3)

with

$$k c = (\omega^2 - \omega_s^2)^{1/2} \text{sign} \Re \omega,$$

$$R_i^* = |r - r_i(t)|.$$

In the wave zone there results from

$$|r - r_i| \approx r - n \cdot r_i, \quad n = r/r, \quad k = kn,$$

(B 4)

$$M_a(\omega, r) = \frac{1}{4\pi c^2} \exp\left\{-i \tilde{k} r\right\} \delta N_a^{(0)}(\omega - \omega_s, \mathbf{k} - \mathbf{k}_s).$$

(B 5)