Calculation of Shock Front Parameters in a Plasma

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Introducing an effective ratio of the specific heats, exact values for the shock front parameters can be obtained, if thermal equilibrium may be assumed. Exact values for the shock front parameters in a singly ionized non-equilibrium plasma can be obtained only if one of the end values is measured.

It is well known how to calculate the parameters (pressure $p_2$, density $\rho_2$, temperature $T_2$, and velocity $v_2$) in the shock heated zone of a gas flow if the specific heats remain constant.1

These calculations, however, become difficult in a plasma flow, where the specific heats (and thus $\gamma = c_p/c_v$) vary $2^{-4}$. In order to use the convenient formalism of gas dynamics some authors 5-7 introduced an effective adiabatic exponent, which we will call $g$. Lunkin 5 (1959) showed, that $g = h/u$, where $h$ is the enthalpy and $u$ the internal energy per gram. $g$ depends on pressure and temperature but, as the variations are only small, it is practice, to estimate $g$ so that one can obtain appropriate values of the shock front parameters in a plasma.

In this paper it is shown, that $g$ can also be used to provide the exact $p_2$, $T_2$, $v_2$, $\rho_2$, $h_2$ in the shock heated plasma, and these quantities can be obtained as tabulated functions of the Mac n number, without any approximations or recursions, provided that thermal equilibrium may be assumed.

In the absence of thermal equilibrium, the end values can be calculated if any one of them is measured. In a non thermal plasma we calculate the ion density $n_i$ rather then the temperature, which is no longer uniquely defined. Deviations from thermal equilibrium are reflected in a change of the magnitude of $g$.

General Shock Equations

The shock front parameter $p_2$, $\rho_2$, . . . produced by a strong shock in a gas flow are in principle determined by the equations of conservation of mass momentum and energy (the frame of reference moves with the shock front)

\begin{align}
q_1 v_1 &= q_2 v_2, \\
p_1 + q_1 v_1^2 &= p_2 + q_2 v_2^2, \\
q_1 h_1 + 1/2 v_1^2 &= q_2 v_2^2 + h_2.
\end{align}

and by the two conditions of state $q = q(p, T)$ and $h = h(p, T)$ if the initial values $p_1$, $q_1$, $v_1$, . . . are specified. The system can be solved analytically in the case of a non reacting gas where $h$ and $q$ are simple functions of $p$ and $T$. In a plasma in thermal equilibrium however $h$ and $q$ depend 8 on the partial pressures of ions, electrons and neutrals which are governed by the Saha equation. Though it is always possible to obtain numerical data for $q$ and $h$, simple analytical equations for enthalpy and density as functions of pressure and temperature can no longer be obtained. Thus it is not possible to solve analytically for the end values in the shock heated zone of a plasma flow. However one can obtain quasi analytical solutions introducing $\gamma = h/u$ to replace the strongly pressure and temperature dependent enthalpy. Fig. 1, 2, 3 show $g(p, T)$ for argon, helium and hydrogen in thermal equilibrium. (It should be noted, that $g$ is also defined in a non thermal plasma — see below.) With the thermodynamical relationship $u = h - p/q$ we obtain from Eq. (4)

\begin{equation}
h_2 = [g_2/(g_2 - 1)] p_2 / q_2.
\end{equation}

2 W. Fickos and J. Arntmann, Z. Physik 172, 118 [1963].
where \[ M_1 = v_1/c_1 = v_1 \cdot \sqrt{\gamma p_1}; \]
and
\[ \gamma = c_p/c_v \text{ (cold gas) and} \]
\[ \epsilon = \frac{2(g_2+1)(\gamma-g_2)}{(\gamma-1)(g_2-\gamma M_1^2)^2}. \]

The minus sign gives the identity while the plus sign represents the shock solution for supersonic flow. With \( M_1 \geq 5 \) (strong shock) and any allowed value of \( g_2, \epsilon \) is small as compared with 1, so that the general shock equations become

\[ q_1/q_2 = v_2/v_1 = (g_2-1)/(g_2+1), \]
\[ p_2/p_1 = 2\gamma M_1^2/(g_2+1) \]

and employing (5) \( h_2 = [2g_2/(g_2+1)] v_1^2 \). (10) For an approximate determination of the end values one can guess \( g_2 \) and solve directly for the end values \( q_2, p_2, v_2, h_2 \). For an exact calculation, however, \( g_2 \) has to be treated as an unknown.

**Thermal Equilibrium**

The plasma is in local thermal equilibrium if electrons, ions and neutrals each have a Maxwell velocity distribution characterized by the same temperature \( T_e = T_i = T_0 = T \), and if the number of ions can be calculated by the Saha equation governed by the same temperature \( T_{\text{Saha}} = T \). For the energy consideration, however, it is not necessary to know, whether the excited states obey a Boltzmann distribution, as the correlated energy term is neglected (see below).

If the shock front in the plasma is supposedly in thermal equilibrium and \( M > 3 \), we have 5 equations: (8) counting twice, (9) and the functions of state \( q(p,T) \) and \( g(p,T) \). These determine the 5 unknowns \( q_2, p_2, g_2, T_2 \) and \( v_2 \). Equilibrium values for \( q(p,T) \) and \( h(p,T) \) and thus

\[ g(p,T) = h/(h-p/q) \]

are tabulated for many gases over a large variety of pressures and temperatures. Using these t-a-

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bles the end values can easily be calculated as function of the Mach number.

We select a pressure \( p_2 \) and solve (8) graphically for this pressure \( p_2 \). This is possible because we can plot both the right side
\[
\frac{g_2 - 1}{g_2 + 1} = \frac{\{g_2(p_2) - 1\}}{\{g_2(p_2) + 1\}}
\]
and the left side
\[
\frac{\varphi_1}{\varphi_2} = \frac{\varphi_1}{\varphi(p_2)}
\]
as function of temperature, taking \( \tilde{p}_2 \) as a constant parameter, Fig. 4. The intersection of the curves satisfies Eq. (8) for the pressure \( p_2 \). The abscissa of the intersection point therefore gives the temperature \( \tilde{T}_2 \) associated with \( \tilde{p}_2 \) while the ordinate \( \varphi \) gives \( \varphi_2 = \varphi_1/a \) and \( \tilde{g}_2 = (1 + a)/(1 - a) \), as \( \tilde{g}_2 \) now is known we can solve (9) for \( \tilde{M}_1 \) thus determining the initial velocity \( \tilde{v}_1 \) which is required to reach this combination of end values. Finally \( v_2 \) can be calculated with (8). These shock front parameters can be plotted as a function of the Mach number. In order to get more points in these graphs, one has to start the calculations with different pressures \( p_2 \). It will be noticed that every point obtained with this procedure is an exact value and no iteration is required. Fig. 5, 6, 7 show an example. \( \varphi_2/\varphi_1, p_2/p_1 \) and \( T_2 \) are given as functions of the Mach number for a strong shock in argon \([h(p_1, T_1) = 0]\) from \( 10 \). Parameter is the initial pressure \( p_1 \). Compression-ratio, Fig. 5, and temperature, Fig. 6, are very different from the values for a non reacting gas with \( \gamma = 5/3 \), and depend on the initial pressure \( p_1 \). The pressure ratio, Fig. 7, does not deviate very much from the inert gas curve \( (\gamma = 5/3) \), and shows so little variation with the initial pressure, that \( p_1 = 10^{-2} \) atm and \( p_1 = 10^{-3} \) atm give identical curves in the scale of Fig. 7.
Deviations from Thermal Equilibrium

In the absence of thermal equilibrium we have to check whether the equations of our system are still valid, and for this end we have to consider how the properties of state depend on the composition of the plasma. For simplicity we consider a single ionized plasma composed of electrons, ions and neutrals (subscripts e, i, o). As well known, the total pressure \( p \) is defined as the sum of the partial pressures

\[
p = p_i + p_e + p_0.
\]

The density is given by

\[
\rho = \rho_i + \rho_e + \rho_0 = n_i m_i + n_e m_e + n_0 m_0
\]

where \( n \) is the number density and \( m \) the mass of the particle. The enthalpy

\[
h = (\rho_e/\rho) h_i + (\rho_e/\rho) h_e + (\rho_0/\rho) h_0
\]

is composed of the partial enthalpies \( h_i, h_e, h_0 \) according to the relative abundance of species

\[
c_i = n_i m_i / \sum j n_j m_j.
\]

If we assume that ions, electrons and neutrals each have a Maxwell velocity distribution, so that we may attribute temperatures \( T_i, T_e, T_0 \) (which may differ) to each of the species, then the partial enthalpies can be written (\( E_i \) = ionization energy)

\[
\begin{align*}
h_i &= \frac{3}{2} k T_i / m_i + E_i / m_i, \\
h_e &= \frac{3}{2} k T_e / m_e, \\
h_0 &= \frac{3}{2} k T_0 / m_0.
\end{align*}
\]

We have neglected the logarithmic term, which means we neglected the net excitation energy compared with the total thermal and ionization energy. The enthalpy then becomes:

\[
h = (1/\rho) \left[ n_i E_i + \frac{3}{2} (n_i k T_i + n_e k T_e + n_0 k T_0) \right]
\]

and with \( n_j k T_j = p_j \) we obtain

\[
h = (1/\rho) \left[ n_i E_i + \frac{3}{2} p_i \right], \quad (11)
\]

similarly the internal energy is defined by

\[
u = (1/\rho) \left[ n_i E_i + \frac{3}{2} p \right]
\]

and thus

\[
g = \frac{h}{u} = \frac{n_i E_i + \frac{3}{2} p}{n_i E_i + \frac{3}{2} p}, \quad (12)
\]

All the variables involved in the three equations of conservation (1), (2) and (3) are defined. Furthermore the thermodynamical relationship \( u = h - p/\rho \) is valid, so that the definition \( g = h/u \) is equivalent to stating

\[
g = \frac{h}{(g - 1) \cdot p/\rho}.
\]

Consequently (8), and (9) are valid in the non thermal plasma. However, the only equation of state available is (12), which gives \( g(p, n_i) \) and does not include undefined temperature. Thus we have only four equations (8) counting twice, (9) and (12) — to determine 5 shock-front parameters \( p_2, \rho_2, v_2, g_2, n_i \). This means nature has freedom to chose a state of non-equilibrium in accordance with the boundary conditions implied by the experiment. In other words, different experiments will produce different forms of non equilibrium. But if we can measure any one of the shock front parameters then all others can be calculated analytically. Suppose, for example, the ratio

\[
v_2/v_1 = \rho_1/\rho_2 = a
\]

was measured and all the initial values (index 1) are known, then we get

\[
\begin{align*}
g_2 &= (1 + a)/(1 - a), \\
p_2 &= p_1 (1 - a) \gamma M_i^2 \\
n_i &= p_1 (1 - a) (1 - 4 a) \gamma M_i^2/2 a E_i.
\end{align*}
\]

Furthermore, if required, \( h_2 \) can be determined from (11). A similar solution is possible, if \( p_2 \) or \( n_i \) is measured.

It might be of interest to know in which direction the shock front parameters will tend to change, if the plasma deviates from thermal equilibrium. We have to distinguish between an overpopulation \( n_i > n_i, \text{saha} \) (frozen ionization) and an underpopulation of ions \( n_i < n_i, \text{saha} \). Differentiating \( g \) in (12) with respect to \( n_i \) we get

\[
\frac{\partial g}{\partial n_i} = -p E_i (\frac{3}{2} p + n_i E_i)^{-2} < 0.
\]

The right side is always negative. Thus an overpopulation will reduce the effective \( g \), while an underpopulation of the ions will raise \( g \). If we apply this information to Eqs. (8) and (9), we see that with \( n_i > n_i, \text{saha} \) the pressure increase \( p_2/p_1 \) and the compression ratio \( \rho_2/\rho_1 \) both will be larger than in thermal equilibrium.

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